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Combinatorial Property of Sets of Boxes in Multidimensional Euclidean Spaces and Theorems in Olympiad Tasks

Pavel S. PANKOV[1], Azret A. KENZHALIEV[2]
[1]*International University of Kyrgyzstan*
[2]*American University of Central Asia*
e-mail: pps5050@mail.ru, azret.kenzhaliev@gmail.com

Abstract. Theorems (in general sense) are constituents of inventing, analysing and solving olympiad tasks. Also, some theorems can be proved with computer assistance only. The main idea is (human) reducing of primary (unbounded) set to a finite one. Non-trivial immanent properties of mathematical objects are of interest because they can be considered as alternative definitions of these objects revealing their additional features. A non-formal indication of such property is only inital data (size of domain) and only output data (proven/not proven) in a corresponding algorithm. One new and two known examples of such properties are considered, some techniques to convert theorem-proving algorithms into olympiad tasks are proposed.

Keywords: olympiads in informatics, immanent property, task, theorem, unboundedness.

1. Introduction

The aim of this paper is to propose computer-assisted search and proof of immanent properties of mathematical objects and to use such theorems in developing of olympiad tasks in informatics.

Theorems (in general sense, statements seemed to be true) are constituents of inventing, analysing and solving olympiad tasks. While authors of olympiad tasks are to describe all statements used for substantiation of their possible solutions, we suppose that contestants also think by means of some statements passing swiftly. We consider this item in Section 3.

Some well-known theorems were discovered by means of computational experiments or can be proved with computer assistance only. The main idea in such proofs is (human) reducing of a primary (unbounded) set to a finite one. We recount one new and one known examples of such theorems and hypotheses for Euclidean spaces of higher dimensions in Section 2.

Non-trivial immanent properties of mathematical objects are of interest because they can be considered as alternative definitions of these objects revealing their additional features. We hope that examples in Section 2 present such properties of Euclidean spaces. We propose the following non-formal indication of such property: an algorithm to prove it or any its corollary has none or only inital data (size of domain) and only output data (*proven/not proven*).

Some techniques to convert theorem-proving algorithms into olympiad tasks are proposed in Section 4.

2. Two Theorems on Immanent Properties of Euclidean Spaces with Unbounded Objects

We will consider Euclidean spaces R^N (N is a natural number) and “boxes” (parallelepipeds parallel to axes).

There are many results on “linear configurations” of finite sets of points on R^2 (see survey, Gardner, 1988, chapter 22), each of them can be considered as a theorem and an immanent property of a plane but they contain vast numerical conditions and are not “unique”.

Buddhist thangkas which do not “use” but “create” linear relations in planar finite sets of points can also be considered as revealing immanent properties of a plane but they are too complex.

We hope that the following problems are “natural” (Pankov, 2008) or have “short and elegant formulation” (Dagienė *et al.*, 2007).

We (Pankov *et al.*, 2005) put the problem on affine configurations without given quantities:

Problem 1. A finite set M is defined as follows (let its points be called M-points):

1) If two segments with endpoints being M-points have only mutual point then it is an M-point.

   This condition is equivalent to the following (let convex hulls of non-empty subsets of M be called HM-sets).

1’) If the intersection of two HM-sets is not empty then it is an HM-set.

2) The set M with any more point does not fulfill the condition 1 (1’).

How many points can such set in R^N ($N \geq 2$) contain?

Consider the plane R^2. As M is finite, there is a “basic” triangle which contains only three M-points (vertices). Exterior of such triangle consists of twelve plane sets: six rays and six infinite domains. Three of these domains cannot contain M-points obviously. Analysis of other nine sets is too complicated but the number of all possible cases is finite. We wrote an interactive program and proved that there exists only essential configuration and

Theorem 1. 1) The answer to Problem 1 in R^2 is only 6. 2) The configuration is the following: three points A, B, C and three points: B’ on prolongation of the segment AB; C’ on prolongation of the segment BC; A’ on prolongation of the segment CA.

Also, we could construct a space model of an M-set of 8 points.

Hence, we put

Hypothesis 1. The space R^N has the immanent "finite-convex-hull"-number = $2N + 2$, $N \geq 2$.

The following statement would facilitate dynamical programming for sets of boxes.

Hypothesis 2. A set of $(N + 1)$ non-overlapping boxes in R^N can be separated by a coordinate hyper-plane (of dimension $(N - 1)$).

This is obvious for $N = 1$ and $N = 2$. Also, there are obvious examples stressing essentiality of this hypothesis:

Example 1 of four square boxes which cannot be separated;

Example 2 of three squares which cannot be separated by a straight line.

Hypothesis 2 seems to be too difficult to be proven for $N = 3$. Hence, we tried to involve computer.

To use computer successfully for proving theorems it is necessary to reduce a task to a finite search (see, for example, Pankov *et al.*, 2012).

Firstly, consider four equal cubic boxes in R^3.

i) There are only two essential alternatives: projections of two cubes onto a coordinate plane are either overlapping or non-overlapping.

Hence, the task is reduced to consideration of integer cubic boxes with sides 2.

ii) Obviously, if any cube is far from others then a separating coordinate plane exists.

Specify this statement.

Lemma 1. If the convex hull of a projection of four integer cubic boxes with sides 2 onto a coordinate (for instance, "vertical") axis is greater than 6 then a separating ("horizontal") plane exists.

Proof. If this convex hull is greater than 6 then the gap between the projections of the "upper" and the "lower" cubes is greater than 2. If projections of two "intermediate" cubes do not fill the gap completely, then a separating plane exists; otherwise: if these projections overlap then a separating plane passes either over or under them otherwise between them.

Hence, it is sufficiently to consider arrangements of four cubes within a cube with side 6.

Such examination (of about 9 million arrangements, see Program 1 `https://cloud.mail.ru/public/MHLv/ktKFSxZ5H`) proved

Theorem 2. A set of 4 non-overlapping integer cubic boxes with side 2 within a cubic box with side 6 can be separated by a coordinate plane.

Applying Lemma 1 we obtain

Theorem 3. A set of 4 non-overlapping equal cubic boxes in R^3 can be separated by a coordinate plane.

This theorem corroborates Hypothesis 2.

By means of improving i) and ii) choppings-off there can be considered four non-equal cubes and general boxes in R^3 and five equal hyper-cubes in R^4.

In other words, Hypothesis 2 may be reformulated as follows:

The space R^N has the immanent "separable-boxes"-number = $N + 1$, $N \geq 1$.

3. Theorems Related to Olympiad Tasks in Informatics

Such mathematical results represented as theorems can be classified as follows:

- Theorems invented or recollected to solve or to facilitate solving of the task (such as Lemma 1 above).
- Theorems proven by means of computer programs written for the task.

In their turn, theorems used by authors of tasks must be proven strictly to justify the author's solution of the task. Mostly, theorems invented by participants during solving tasks pass swiftly, in implicit form without verbal formulation. It is enough to be assured in their validity for the participant (nevertheless, sometimes is useful to write down any formulation to clarify the participant's thoughts for themself).

Remark. Sufficiency of the participant's conviction on validity of an invented "theorem" depends on conditions of the competition. If results of testing programs are shown immediately to the participant (as it is in use at the ACM-ICPC International Collegiate Programming Contests and it was at National OI in Kyrgyzstan, March 2018) then the participant would submit the program based on this "theorem" without firm conviction; successive passing of all tests proves either validity of such "theorem" or its failing only in very exotic cases which were not covered in the set of tests.

If results of testing programs appear after the contest then the participant would be assured (in any way) in the validity of "theorem".

As regards "theorems" to be proven by means of computer programs during a contest. Every correct program solving any correct task can be formally expressed as a "theorem" but with a too vast statement, including mathematical description of the set of initial data etc.

Some techniques to develop olympiad tasks on proving "intensional" theorems are proposed below.

4. Developing of Tasks of Type "to Prove a Theorem"

We will consider this item on examples of Theorem 1 and Theorem 2.

Firstly, one must not propose a task of type „*write a program to prove the statement ...*" or „*write a program to check validity of the statement...*" because the jury would have to check listings of programs submitted what is practically impossible.

Remark. A similar situation is at mathematical olympiads. A common type of tasks is „*to prove the statement ...*" But contestants' solutions of such tasks put a thankless duty for jury involving them into tangle debates and appeals: to prove that a submitted text is not a complete proof (although it certainly contains parts of actual proof). We propose to convert such tasks into quantitative ones, as well as below.

Secondly, in our opinion, it is not convenient to propose tasks with responds of type „*yes*“/“*no*“ because there is probability of partially random guessing.

We propose to develop tasks with vast quantitive respond.

For example, Problem 1 may be put as

Task 1. Given a natural N in 2..10. How many sets M of integer points in the square $[-N..N] \times [-N..N]$ meet the following conditions (let their points be called M-points)?

1) The three points (0,0), (1,0) and (0,1) are M-points.
2) If two segments with endpoints being M-points have only mutual point then it is an M-point.
3) The set M with any more integer point in $[-N-1 .. N+1] \times [-N-1 .. N+1]$ does not meet the condition 1.

Write a program which outputs this number (mod 1000) (as usually, CPU time is 1 second).

Solving for $N = 2$ and $N = 3$ can be made by means of almost full search; solving for $N > 3$ demands improving of search, i.d. elements of proof (in mind) of Theorem 1. (For jury: answer follows immediately from Theorem 2: two options of three rays; only point on each of them).

The general idea of computer proof of a theorem of type (*) “$(\forall x \in X)(P(x))$” where X is an infinite set or a “too vast” one and $P(x)$ is a predicate is reducing (*) to (**) “$(\forall x \in X_1)(P(x))$” where X_1 is a finite set accessible for a computer.

Hence, the following general task for using at contests on programming can be formulated:

How many $x \in X_1$ meet the condition $P(x)$? If the contestant would be able to write a corresponding program then the answer will be: all $|X_1|$. Then they may be congratulated: “You have proven the theorem “$(\forall x \in X_1)(P(x))$” and ipso facto done the general theorem “$(\forall x \in X)(P(x))$” ”.

For example, Theorem 2 (CPU time of Program 1 is about 36 seconds).

Task 2. Given an integer N in 4 .. 6. How many sets of 4 non-overlapping integer cubic boxes with side 2 within a cubic box with side 6 can be separated by a coordinate plane? (CPU time is 1 second).

To obtain full score the participant is to improve Program 1.

Some immanent properties can be also represented as “$(\exists x \in X)(P(x))$” or “calculate *min* (*max*) $\{F(x) : x \in X\}$” with unexpected result.

For example, consider the Simpson’s paradox: there exist such positive integer numbers

(***) $A1 < B1, A2 < B2, A3 < B3, A4 < B4$ that

(****) $A1 / B1 > A2 / B2$ and $A3 / B3 > A4 / B4$ and $(A1 + A3) / (B1 + B3) < (A2 + A4) / (B2 + B4)$.

Task 3 (simple). Given N in 14..100. Find such (***) that (****) and $max\{B1, B2, B3, B4\} = N$.

Task 4. Given N in 14 .. 100. Calculate the common fraction

$max\{min\ \{A1 / B1 - A2 / B2, A3 / B3 - A4 / B4, (A2 + A4) / (B2 + B4) - (A1 + A3) / (B1 + B3)\}:$
$(***)\ , B1 \leq N, B2 \leq N, B3 \leq N, B4 \leq N\}.$

5. Conclusion

We hope that computer-assisted search for immanent properties of mathematical objects would yield new intensional tasks being contributions to the mathematical science too and their solving would be interesting for participants of various contests on informatics and demonstrate them capacities of computers in scientific investigations.

6. Appendix – Task Spear

As gratitude to the hosts of the IOI'2018, we propose the following set of tasks for investigation.

It is known that Japan appeared as Drops into Ocean from Spear.

Let us try to optimize this process.

Task: given a binary matrix ('0's mean Ocean, '1's do Land) and the set of possible steps of Spear.

Initially Spear is over the NE corner of the matrix.

How many steps of Spear are necessary to create all Lands (to pass all '1's ?)

The simplest sufficient set of possible steps is {S, W, E}.

Example: the matrix

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 1 0 0
0 0 0 0 0 0 0 0 0 0 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 1 1 0 0 0
0 0 0 0 0 0 0 0 1 1 0 0 0 0
0 0 0 0 0 0 1 1 1 1 0 0 0 0
0 0 0 1 1 1 1 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 1 0 1 1 0 0 0 0 0 0 0
0 0 0 1 0 0 0 0 0 0 0 0 0 0
```

Possible beginnings of the optimal ways: WWSES… or SWWSE…

The answer is 32.

Until what size of the matrix can you construct an effective algorithm?

What other sets of possible steps ought to be considered (for example {S, SW, SE, W, E})?

What effective algorithms can be developed for such sets?

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P.S. Pankov (1950), doctor of physical-math. sciences, prof., corr. member of Kyrgyzstani National Academy of Sciences (KR NAS), was the chairman of jury of Bishkek City OIs, 1985–2013, of Republican OIs, 1987–2012, the leader of Kyrgyzstani teams at IOIs, 2002–2013. Graduated from the Kyrgyz State University in 1969, is a head of laboratory of Institute of mathematics of KR NAS, a professor of the International University of Kyrgyzstan.

A.A. Kenzhaliev (1999). Bronze medal at IOI'2016. Teacher Assistant at American University of Central Asia. Student of Korea Advanced Institute of Science and Technology (KAIST), Class of 2022.