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Tasks of a Priori Unbounded Complexity

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Abstract. We consider the following types of tasks: (*) Is there an element meeting some conditions in an infinite set? (**) If such element a priori exists, find any element or the first of such elements. By the condition, the number of operations (steps) to solve the task is unbounded, and the first aim is to guess any upper estimation for this number and further to improve this estimation. In general, tasks (*) are non-resolvable because they are reduced to the well-known problem of words identity in Markov algorithmical language; tasks (**) are resolvable because an infinite discrete set is countable. We hope that some classes of such tasks may be of interest for using in informatics olympiads of various levels. A survey of such tasks in various branches of informatics and some ideas to create and to solve them are presented.

Key words: informatics olympiads, tasks, unboundedness.

1. Survey of Types of Tasks and the Aim of Paper

Most of tasks offered at informatics olympiads are such that any (non-effective) algorithm (of exponential or polynomial complexity) and a (rough) estimation of steps during its execution are obvious and this algorithm can be implemented immediately.

Remark 1. We advise participants, who are not so experienced, to implement such an algorithm first and to check better algorithms (with small initial data) with it.

We consider another type of tasks: a priori, the number of operations (steps) to solve the task is unbounded (in other words, the search is in an infinite set). In general, such tasks are non-resolvable because some of them are reduced to the well-known problem of words identity in Markov algorithmical language or to Diophantine equations.

Nevertheless, we hope that some classes of such tasks may be of interest for use in informatics olympiads of various levels. A survey of such tasks in various branches of informatics and some ideas of to solve them are presented. Certainly, the jury is to know existence of a solution and any algorithm which can solve the task in appropriate time. This knowledge is the main information for the participant of the contest.

In general, such tasks may be put as follows:

General task 1. Find an element meeting some conditions in an infinite set.

General task 2. Find the first elements meeting some conditions in an infinite ordered set.

Here are some initial ideas:

Idea 1. Guess any finite set containing (or an upper estimation for) such element and further to improve this estimation.

Idea 2. Reduce the search on the infinite set to one on any sparse subset.

Idea 3. Distinguishing of "basic" or "unit" solutions to build a general solution of them (or an upper estimate for the solution).

In its turn, Idea 1 may be attained by

Idea 1a. Proof or guessing of some periodicity. Then the first period only is to be considered.

Idea 1b.Using the least common multiple for proof of periodicity.

To generate such tasks, the following ways are proposed:

Way 1. Use any known mathematical result (in number theory, theory of finite groups, combinatorics, graph theory etc.). Some examples are given below. It may be used for training but not for high level olympiads.

Way 2. Set an interesting task and try to solve it yourself.

Way 3. Take (occasionally) any element or notice any interesting element of the finite set, formulate its properties and try to construct an algorithm which can find this element in appropriate time.

Remark 2. Tasks of search on a finite set (easier) also can be constructed in such a way. After defining the element to be found we announce any a priori upper boundary greater than this element.

2. Number Theory

Let us begin with classical results (Way 1). All numbers are assumed to be integer and non-negative.

Task 3 (Chinese theorem of residues). Given prime numbers $p_1 < p_2 < \ldots < p_n$ and numbers $r_1 < p_1, r_2 < p_2, \ldots r_n < p_n$, find a number K such that $K \bmod p_i = r_i$, $i = 1..n$.

First step of solution (Idea 1b). If K meets these conditions then $K \pm p_1 p_2 \ldots p_n$ meets them too. Hence, to check numbers $1..p_1 p_2 \ldots p_n$ is sufficient ($O(p_1 p_2 \ldots p_n)$ operations).

An effective algorithm is also obvious: Let $K_1 := r_1$. Find such $d_{i+1}(0 \leqslant d_{i+1} < p_{i+1})$ that

$(p_1 p_2 \ldots p_i d_{i+1} + r_i) \bmod p_{i+1} = r_{i+1}$ and let $K_{i+1} = p_1 p_2 \ldots p_i d_{i+1} + r_i$, $i = 1..n-1$.

($O(p_1 + p_2 + \ldots + p_n)$ operations).

Task 4 (amicable numbers). Find such numbers u and v that the sum of proper divisors of u equals v and the sum of proper divisors of v equals *u*. (The first pair is small: 220 and 284, so an interesting task is the following: Find two pairs...).

Adding numbers themselves to the list of divisors we obtain a more convenient formulation:

Find such numbers u and v that the sums of their divisors equal $(u + v)$; and the formula for solving: sum of divisors of a number with prime factor decomposition $p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$ is

$$(1 + p_1 + \dots + p_1^{k_1})(1 + p_2 + \dots + p_2^{k_2}) \dots (1 + p_n + \dots + p_n^{k_n}).$$

Task 5 (confutation of Fermat's hypothesis).

Find the least number t such that the number $2^{2^t} + 1$ is not prime.

Task 6 (confutation of Euler's hypothesis).

a) Find the least number t such that there exists a solution of the equation $x^4 + y^4 + z^4 = t^4$ in positive numbers.

b) Find the least number u such that there exists a solution of the equation $x^5 + y^5 + z^5 + t^5 = u^5$ in positive numbers.

To solve this task successfully the technique of omitting the inner cycle is applicable.

Task 7 (Ramanujan's problem).

Find the least number t such that there exists more than one solution of the equation $x^3 + y^3 = t$ in positive numbers.

Way 3: (the simplest example) calculate powers of natural numbers and combine them to obtain small numbers. Maybe, we have found the equality $984^2 + 2^{12} = 5 + 99^3$. Thus, we have obtained the following tasks:

Task 8. Find a solution of the equation in positive numbers: a) $x^2 + y^{12} = 5 + z^3$; b) $x^2 + y^{12} = u^2 + 1 + z^3$; c) $x^2 + y^{12} = u^2 + w^4 + z^3, \dots$

If the jury finds an effective algorithm to solve any of these tasks then this task may be proposed at an olympiad.

3. Geometry

Task 9 (classical). One can jump F feet forward and B feet backward along the straight line (one-dimensional grid). How many steps (at least) are necessary to reach the point located X feet from the initial one?

Idea 3. Certainly, if X is not divisible by $G := GCF(F, B)$ then the task has no solution. Otherwise, solve the Diophantine equation $F/G * U - B/G * V = 1$ in positive numbers (the most effective algorithm for this task is using continuous fractions). Then the answer is not greater than $X/G * (U + V)$.

A similar task for a two-dimensional grid is more interesting.

Task 10. A hare is at the origin of coordinates and can jump only to integer points (with integer coordinates). Each of its jumps is of length 5. How many jumps (at least) are necessary to reach a given point *(X, Y)*?

Idea 3. Make some attempts. Jumps *(3,4)* and $(3, -4)$ result *(6,0);* shift *(6,0)* and jump $(-5, 0)$ result shift *(1,0)* and the task is reduced to Task 9.

Many tasks can be put on configurations (see Gardner, 1988, chapter 22). To make tasks more determined, we will consider configurations on grids only.

Way 3. Choose any finite set on a two-dimensional grid. Some subsets of (more than 2) points lie along straight lines (rectilinear subsets). Take into account cardinal numbers of these subsets. We obtain the following task on affine configurations.

Task 11. Find any (or with the least cardinal number), (or of the least size) configuration having prescribed numbers of rectilinear subsets of prescribed cardinal numbers.

Way 3. Choose any finite set on a two-dimensional grid. Compute squares of distances between pairs of points within the set and detect numbers representing equal distances. We obtain the following task on metric configurations.

Task 12. Find any (or of the least size) configuration such that values of distances between its points form a set of a prescribed cardinal number and numbers of distances of same value are also given. (Values themselves are not given; otherwise any a priori upper estimation of size of the set can be derived).

Remark 3. Some tasks on configurations arise from arrangements of stars on the flag of the United States (see Gardner, 1988, chapter 22).

Remark 4. Triangular and hexagonal grids may also be involved.

The following unique task on affine configurations without given quantities was discovered by S.N. Alekseenko (Pankov *et al.*, 2005).

Task 13. Find any configuration (set) of points fulfilling the following conditions:

A1) If two segments with ends belonging to the set have only one mutual point then it belongs to the set too.

A2) No point can be added to the set, preserving the property A1).

4. Acceleration of Algorithms

Most of tasks listed above can be transformed to the following type: given a (slow) algorithm. Write a program yielding the same result in appropriate time. For instance, Task 6b:

$u := 0$; *equal := false*;
Repeat
$u := u + 1$;
for $x := 1$ *to* u *{ for* $y := 1$ *to* u *{ for* $z := 1$ *to* u *{ for* $t := 1$ *to* u
{if $x^5 + y^5 + z^5 + t^5 = u^5$ *then {equal := true;* $x_1 := x$*;* $y_1 := y$; $z_1 := z$; $t_1 := t$ *}*
}}}}
until equal;
Output x_1, y_1, z_1, t_1, u.

5. Conclusion

We hope that successful application of methods proposed above would yield new tasks with ”short and elegant formulation” (Dagiene *et al.*, 2007); solving of such tasks created by Way 3 would yield interesting ”minimal” objects.

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