COMBINATORIAL PROPERTY OF SETS OF BOXES IN MULTIDIMENSIONAL EUCLIDEAN SPACES AND THEOREMS IN OLYMPIAD TASKS

Pavel S. Pankov, Azret A. Kenzhaliev

Tsukuba -2018

Abstract. Theorems (in general sense) are constituents of inventing, analysing and solving olympiad tasks. Also, some theorems can be proved with computer assistance only. The main idea is (human) reducing of primary (unbounded) set to a finite one. Non-trivial immanent properties of mathematical objects are of interest because they can be considered as alternative definitions of these objects revealing their additional features. A non-formal indication of such property is only inital data (size of domain) and only output data (proven/not proven) in a corresponding algorithm. One new and two known examples of such properties are considered, some techniques to convert theorem-proving algorithms into olympiad tasks are proposed.

1. Theorems everywhere

Statements which can be considered as "theorems" arise in various branches of informatics and mathematics permanently. Sometimes they are not called "theorems".

For example, "$\sqrt{2} \approx 1.41$" is not a theorem, but "$|\sqrt{2} - 1.41| < 0.01$" or "$\sqrt{2} \in [1.40; 1.42]$" are theorems.

Any result of computation by a program involving symbols and integer numbers (discrete objects) may be interpreted as a theorem.

Remark. A result of approximate computations by a program involving "real" (floating-point) numbers may be interpreted as a theorem only if all rounds off are outward (strict cumulative estimation of all rounds off is practically impossible).

2. Two theorems on immanent properties of Euclidean spaces with unbounded objects

We will consider Euclidean spaces R^N, "boxes" (parallelepipeds parallel to axes), and finite sets of points.

There are many results on "linear configurations" of finite sets on R^2, each of them can be considered as a theorem and an immanent property of a plane but they contain vast numerical conditions and are not "unique".

Buddhist thangkas which do not "use" but "create" linear relations in finite sets on R^2 can also be considered as revealing immanent properties of a plane but they are too complex.

We hope that the following problems are "natural" (Pankov, 2008) or have "short and elegant formulation" (Dagiene et al., 2007).

Problem 1. A finite set M is defined as follows:

1) If two segments with end-points being M-points have only mutual point then it is an M-point.

2) The set M with any more point does not meet the condition 1.

How many points can such set in R^N $(N \geq 2)$ contain?

For $N=2$ there is a "basic" triangle which contains only three M-points (vertices). Analysis of other nine sets is too complicated but the number of all possible cases is finite. We wrote an inter-active program and proved that there exists only essential confi-guration and

Theorem 1. The answer to Problem 1 in R^2 is only 6. Three col-linear triples: ABC'; BCA'; CAB'.

Hypothesis 1. The space R^N has the immanent "finite-convex-hull"-number = $2N+2$, $N \geq 2$.

The following statement would facilitate dynamical programming for sets of boxes.

Hypothesis 2. A set of $(N+1)$ non-overlapping boxes in R^N can be separated by a coordinate hyper-plane (of dimension $(N-1)$).

This is obvious for $N=1$ and $N=2$ and seems to be too difficult to be proven for $N=3$. Consider a particular case - four equal cubic boxes in R^3. Reduce this task to a finite search.

i) There are only two essential alternatives: projections of two cubes onto a coordinate plane are either overlapping or non-overlapping. Hence, the task is reduced to consideration of integer cubic boxes with sides 2.

ii) Obviously, if any cube is far from others then a separating coordinate plane exists.

Specify this statement.

Lemma 1. If the convex hull of a projection of four integer cubic boxes with sides 2 onto a coordinate (for instance, “vertical”) axis is greater than 6 then a separating (“horizontal”) plane exists.

Hence, it is sufficiently to consider arrangements of four cubes within a cube with side 6. Such examination (of about 9 million arrangements, see Program 1

https://cloud.mail.ru/public/MHLv/ktKFSxZ5H) proved

Theorem 2. A set of 4 non-overlapping integer cubic boxes with side 2 within a cubic box with side 6 can be separated by a coordinate plane. Applying Lemma 1 we obtain

Theorem 3. A set of 4 non-overlapping equal cubic boxes in R^3 can be separated by a coordinate plane.

This theorem corroborates Hypothesis 2.

3. Types of theorems related to Olympiad tasks in informatics

- Theorems invented or recollected to solve or to facilitate solving of the task (such as Lemma 1 above).

- Theorems proven by means of computer programs written for the task.

In their turn, theorems used by authors of tasks must be proven strictly to justify the author's solution of the task. Mostly, theorems invented by contestants during solving tasks pass swiftly. It is enough to be assured in their validity for the contestant (nevertheless, sometimes is useful to write down any formulation to clarify the contestant's thoughts for themself).

Sufficiency of the contestant's conviction on validity of an invented "theorem" depends on conditions of the competition.

If results of testing programs are shown to the contestant immediately (for instance, the competitions ACM-ICPC, National OI in Kyrgyzstan, 2018) then the participant would submit the program based on this “theorem” without firm conviction.

If results of testing programs appear after the contest then the participant would be assured (in any way) in the validity of “theorem”.

Some techniques to develop olympiad tasks on proving “intensional” theorems are proposed below.

4. Developing of tasks of type “to prove a theorem”

We will consider this item on examples of Theorems 1 and 2.

Firstly, one ought not to propose a task of type „*write a program to prove the statement* ...“ or „*write a program to check validity of the statement...* “ because the jury would have to check listings of programs submitted what is practically impossible.

Remark. A similar situation is at mathematical olympiads. A common type of tasks is „*to prove the statement* ...“ But contestants‘ solutions of such tasks put a thankless duty for jury involving them into tangle debates and appeals: to prove that a submitted text is not a complete proof (although it certainly contains parts of actual proof). We propose to convert such tasks into quantitative ones, as well as below.

Secondly, in our opinion, it is not convenient to propose tasks with responds of type „*yes*“/“*no*“ because there is probability of partially random guessing.

We propose to develop tasks with vast quantitive respond.

For example, Problem 1 may be put as

Task 1. Given a natural N in 2..10. How many sets M of integer points in the square $[-N..N]\times[-N..N]$ meet the following conditions (let their points be called M-points)?

1) the three points $(0,0)$, $(1,0)$ and $(0,1)$ are M-points;

2) if two segments with endpoints being M-points have only mutual point then it is an M-point;

3) the set M with any more integer point in the square does not meet the condition 1.

Write a program which outputs this number (mod 1000) (as usually, CPU time is 1 second).

The general idea of computer proof of a theorem of type (*) "$(\forall x \in X)(P(x))$" where X is an infinite or a "too vast" set is reducing (*) to (**) "$(\forall x \in X_1)(P(x))$" where X_1 is a finite set accessible for a computer. Hence, the following general task for contests on programming can be formulated: How many $x \in X_1$ meet the condition $P(x)$? If the contestant would be able to write a corresponding program then the answer will be: all $|X_1|$. Then they may be congratulated: "You have proven the theorem (**) and ipso facto done the general theorem (*).

For example, Theorem 2 (CPU time of Program 1 is about 36 seconds): Task 2. Given an integer N in 4..6. How many sets of 4 non-overlapping integer cubic boxes with side 2 within a cubic box with side N can be separated by a coordinate plane? (CPU time is 1 second).

To obtain full score the contestant is to improve Program 1.

Conclusion

We hope that computer-assisted search for immanent properties of mathematical objects would yield new intensional tasks being contributions to the mathematical science too and their solving would be interesting for participants of various contests on informatics and demonstrate them capacities of computers in scientific investigations.

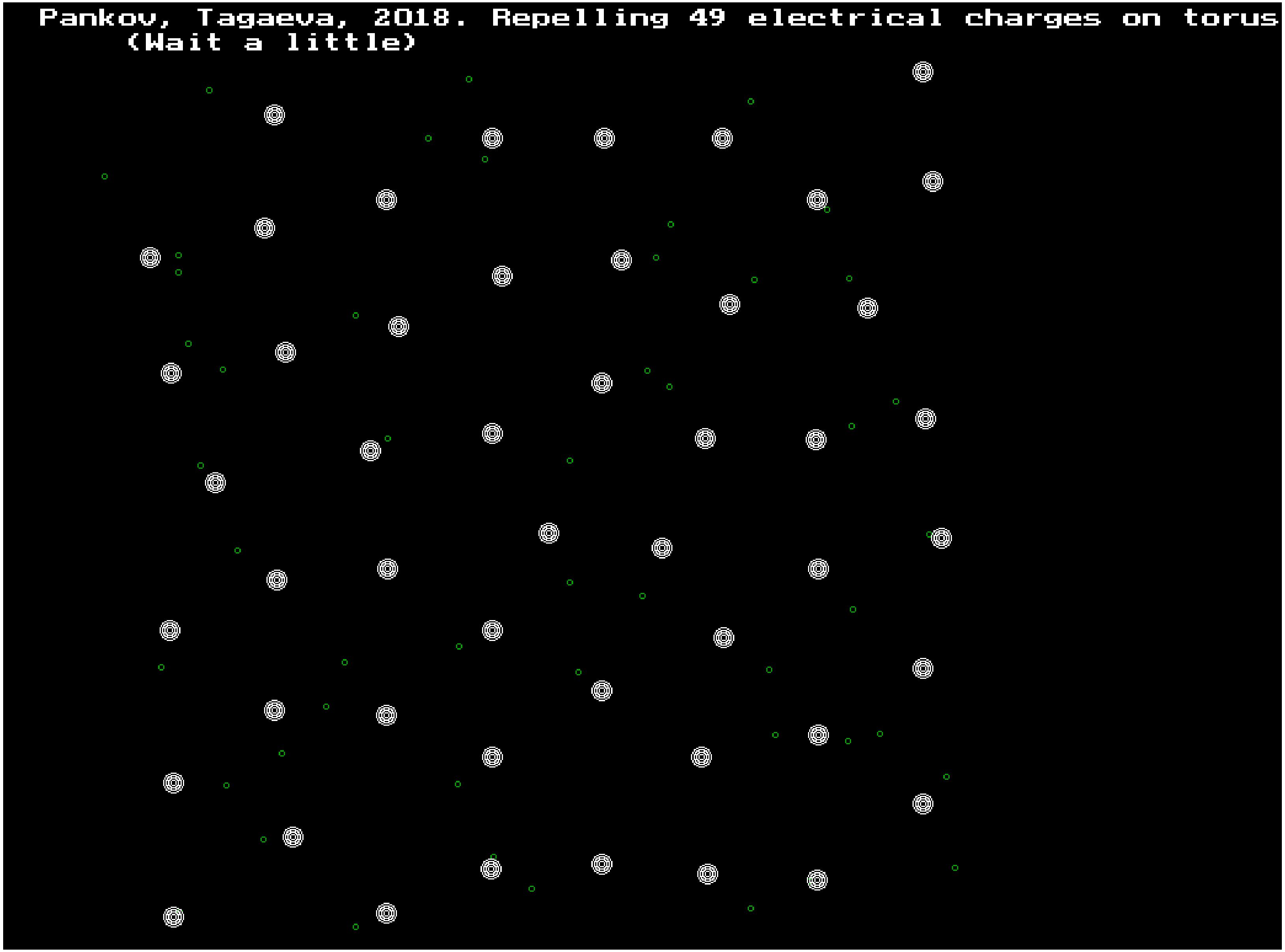
Appendix 1. Phenomenon of formation of a regular grid for many repelling electrical charges on a topological torus

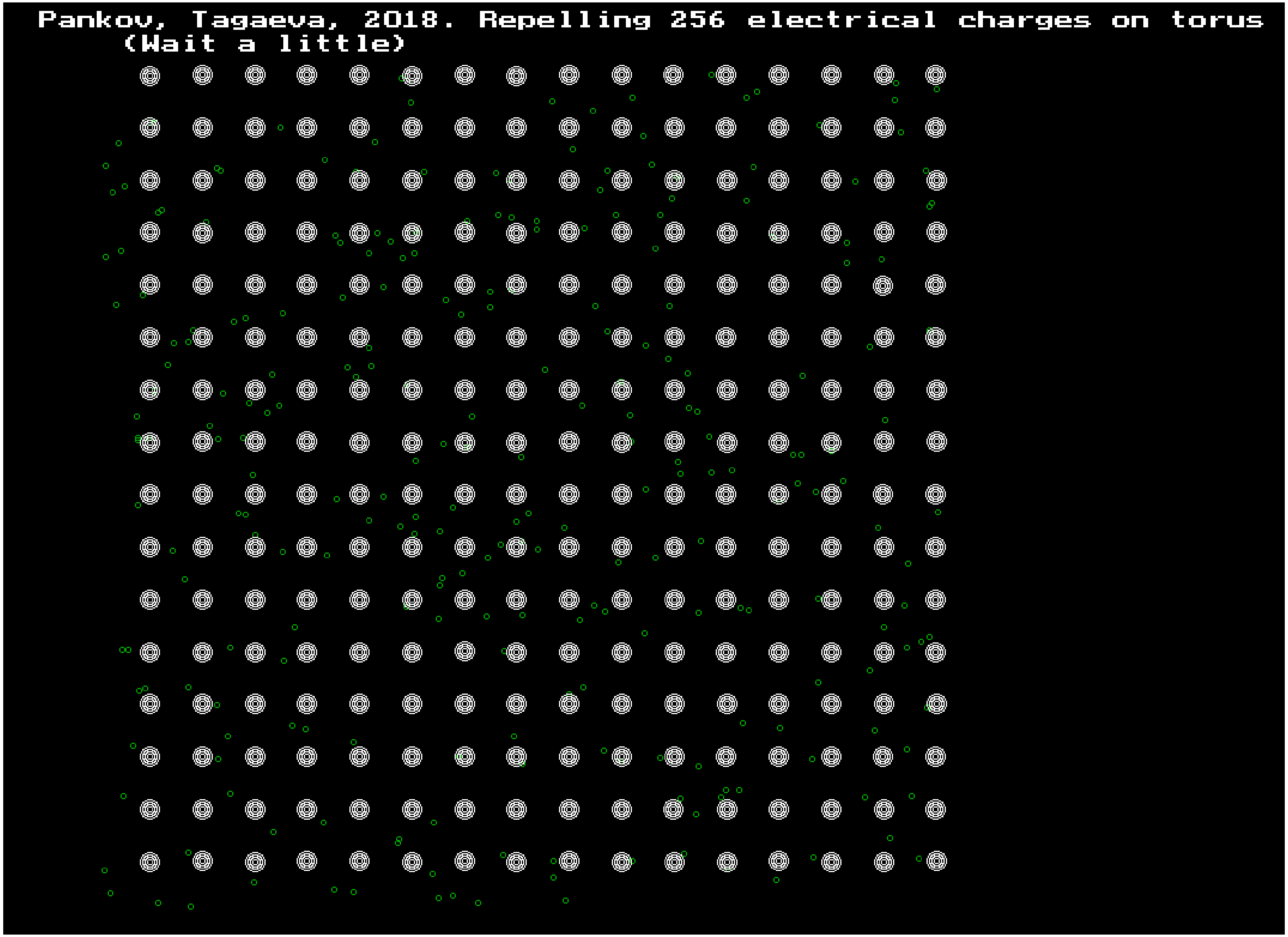
Together with S.Tagaeva, by means of numerical experiments the following regularity was opened.

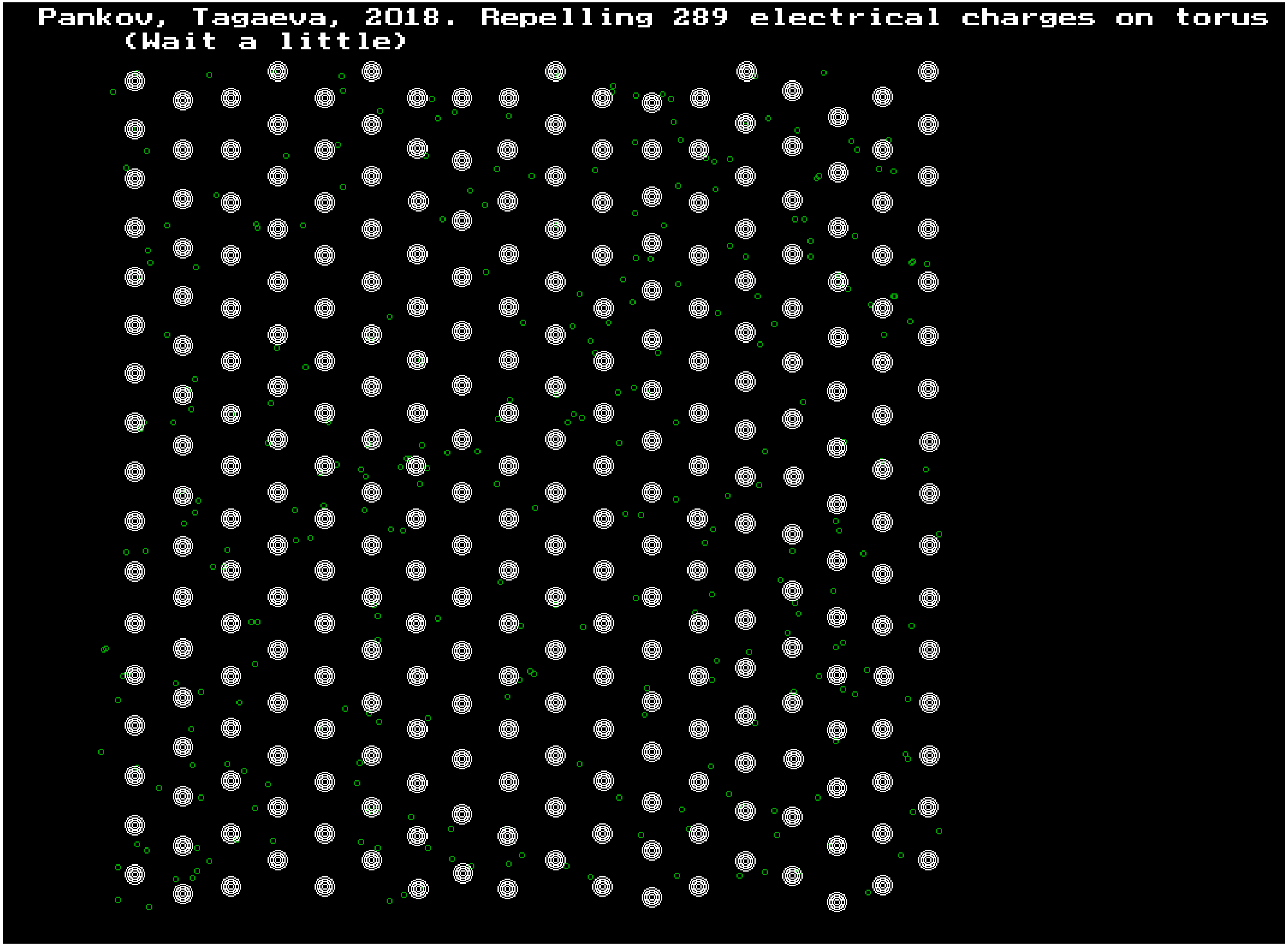
For any random initial distribution of many (N greater than 50) like charges on a topological torus (a square with opposite sides glued) in a viscous media these charges form a final regular grid; if N is a square of even number then the grid is square mainly;

if N is a square of odd number then the grid is triangular mainly.

This is a consequence of the effect of numerousity.


Pankov, Tagaeva, 2018. Repelling 49 electrical charges on torus
(Wait a little)


Pankov, Tagaeva, 2018. Repelling 256 electrical charges on torus
(Wait a little)


Pankov, Tagaeva, 2018. Repelling 289 electrical charges on torus
(Wait a little)

Appendix 2. Task Spear

As gratitude to the hosts of the IOI’2018, we propose the following set of tasks for investigation.

It is known that Japan appeared as Drops into Ocean from Spear.

Let us try to optimize this process.

Task: given a binary matrix (‘0’s mean Ocean, ‘1’s do Land) and the set of possible steps of Spear.

Initially Spear is over the NE corner of the matrix.

How many steps of Spear are necessary to create all Lands (to pass all ‘1’s ?)

The simplest sufficient set of possible steps is {S, W, E}.

Example: the matrix

```
0000000000000
0000000000100
0000000001110
0000000000000
0000000011000
0000000110000
0000011110000
0001111000000
0000000000000
0001011000000
0001000000000
```

Possible beginnings of the optimal ways: WWSES… or SWWSE…

The answer is 32.

Until what size of the matrix can you construct an effective algorithm?

What other sets of possible steps ought to be considered (for example {S, SW, SE, W, E}) ?

What effective algorithms can be developed for such sets?

THANK YOU
FOR ATTENTION!