Tidal Flow: A Fast and Teachable Maximum Flow Algorithm

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Algorithms Live!

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Maximum Flow Problems in CP

- Problems are interesting
- High variation of problem ideas (vs spanning tree algorithms, shortest paths, etc)
- Harder problems require faster algorithms

Timeline for Tidal Flow

2012		ACM ICPC WF (invented)
2013	}	Taught at U Central Florida
...		
2016		
2017		IOI Conference suggested
2018		Present

Algorithms in Competitive Programming

- Easy to teach
- Easy to implement
- Efficient in practice

Example: Fenwick Tree

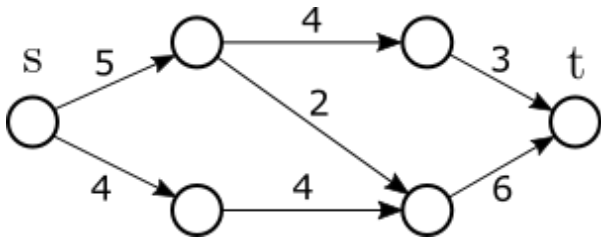
```
void update(int i, int delta) {
   while (i < size) {
      table[i] += delta;
      i += i&-i;
   }
}
```

```
void sum(int i) {
   int sum = 0;
   while (i > 0) {
      sum += table[i];
      i -= i&-i;
   }
   return sum;
}
```

Overview

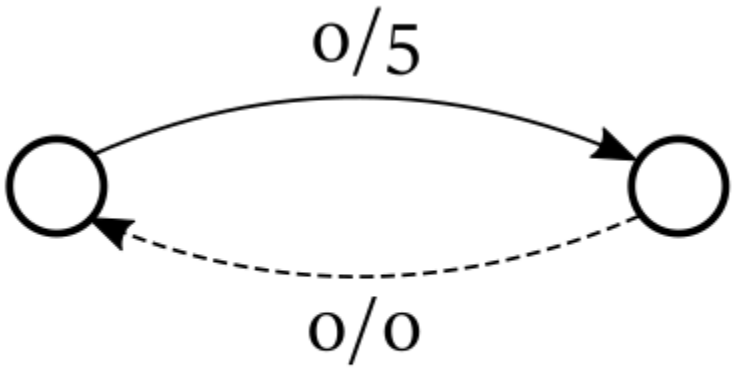
- Review of the maximum flow problem and blocking flows
- Tidal Flow
- Theoretical performance
- Empirical performance
- Open questions

Maximum Flow



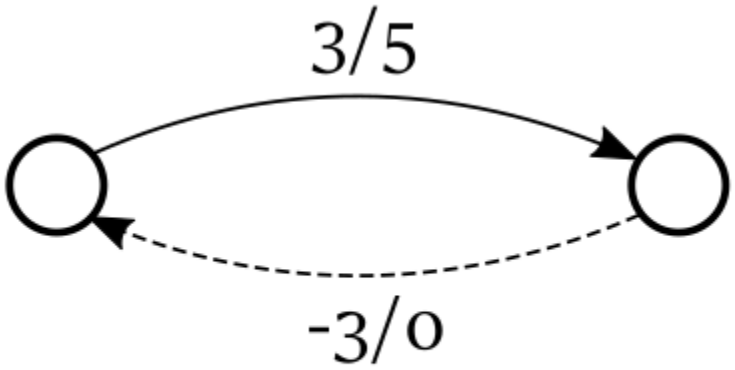
- Input: (G, s, t, cap)
  - $G = (V, E)$
  - $s, t \in V$
  - $cap: E \to \mathbb{R}^+$
- Output: $f: E \to \mathbb{R}^*$
- Constraints:
  - Capacity Constraint: $f(e) \leq cap(e)$ for all $e \in E$
  - Conservation Constraint: $\sum f(w, v) = \sum f(v, u)$ for all $v \in V \setminus \{s, t\}$
- Maximize: $\sum f(s, v)$ for all $v \in V$

Maximum Flow: Reverse Edges



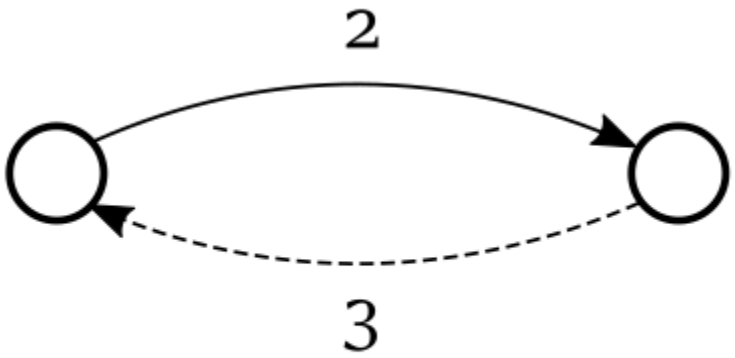
flow/cap

Maximum Flow: Reverse Edges



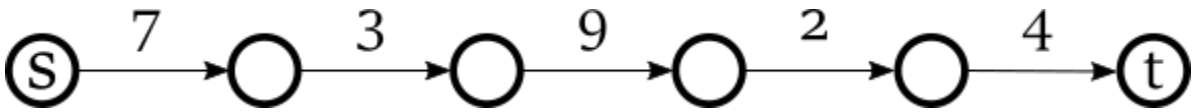
$f(w, v) = -f(v, w)$ for all $(w, v) \in E$

Maximum Flow: Residual Capacities

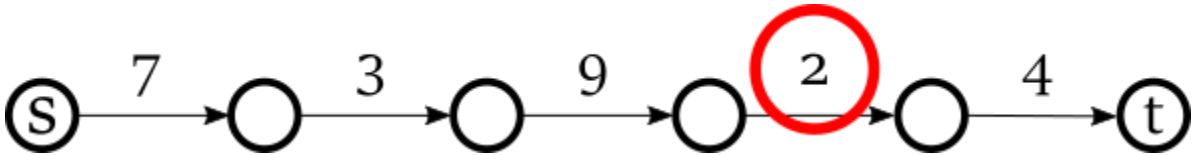


$$cap_r(w, v) = cap(w, v) - f(w, v) \text{ for all } (w, v) \in E$$

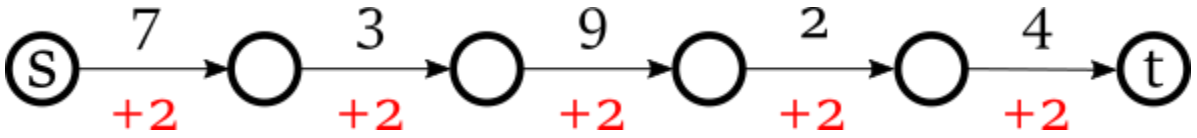
Augmenting Paths



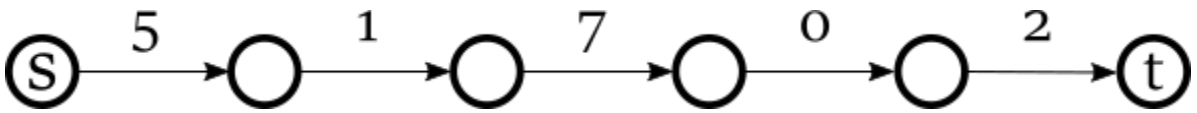
Augmenting Paths



Augmenting Paths



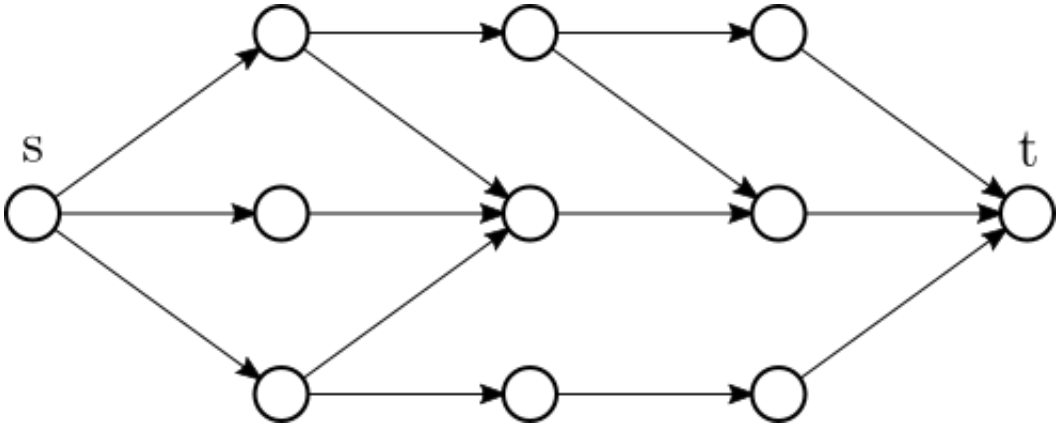
Augmenting Paths



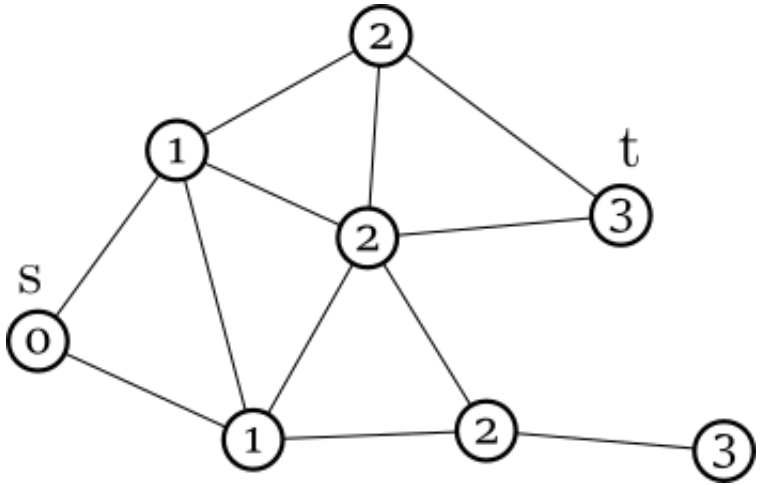
Augmenting Paths: Runtime

- Arbitrary Paths: $O(fm)$
- Shortest Paths: $O(nm^2)$

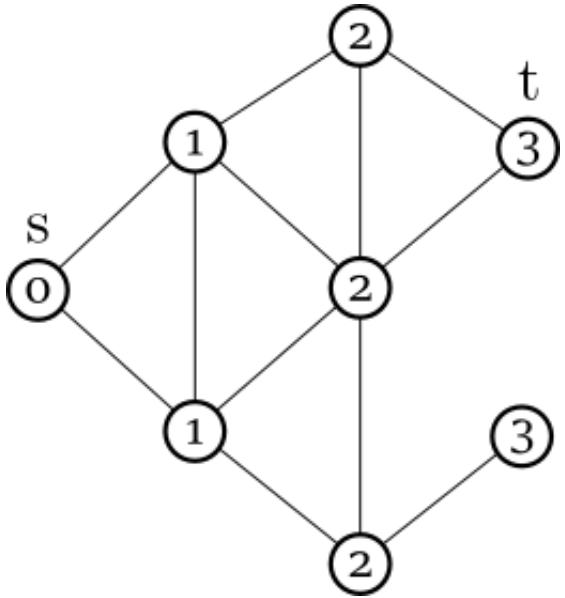
Dinitz Algorithm



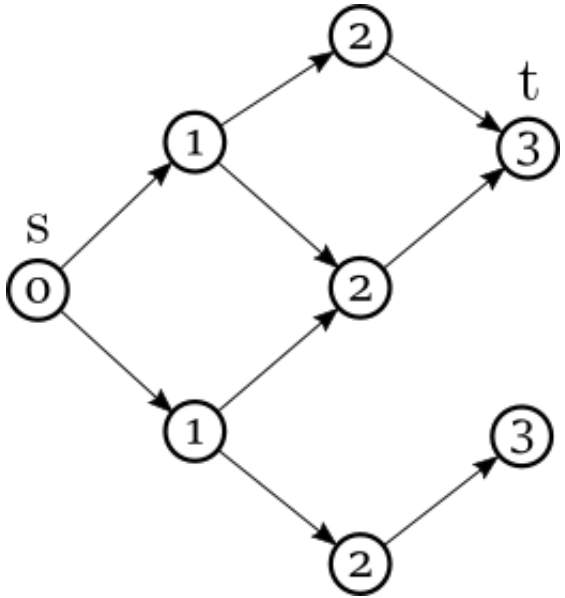
Level Graph



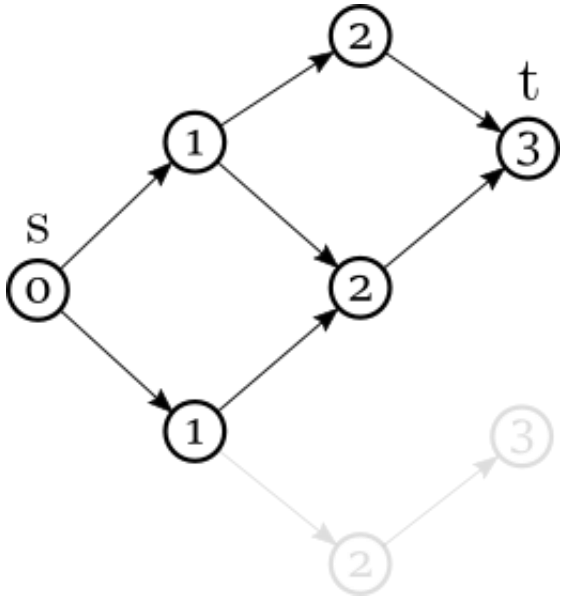
Level Graph



Level Graph



Level Graph



Dinitz

- Blocking flows discovered in $O(nm)$ through modified DFS
- At most $O(n)$ level graphs possible
- Runtime: $O(n^2m)$

Tidal Flow

- Same goal as Dinitz: block the level graph
- Try blocking in $O(m)$ runtime
- Failure is OK! Keep trying.

Tidal Flow: Conceptual Metaphor

“A (conceptual) metaphor is a cognitive process that occurs when a subject seeks understanding of one idea (the target domain) in terms of a different, already known idea (the source domain). The subject creates a conceptual mapping between the properties of the source and the target, thereby gaining new understanding about the target.” (Forišek and Stienová, 2013)

Here the metaphor is oceanic tides.

Tide Cycle

- High Tide
- Low Tide
- Erosion

Tide Phase: High Tide

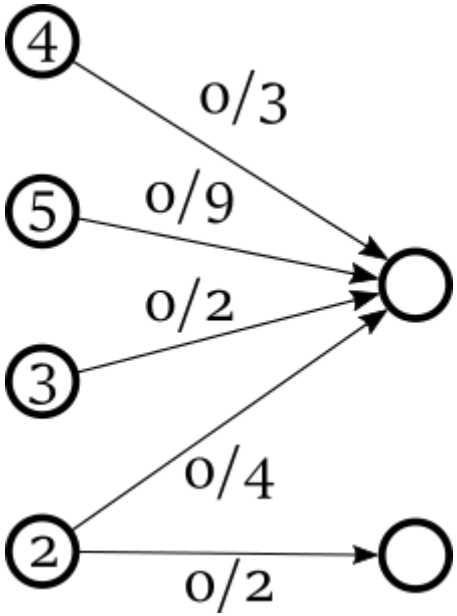
- For each vertex, compute $h: V \to R^*$

$$h(v) = \sum_{w \in G(v)} \min(cap(w,v) - f(w,v), h(w))$$

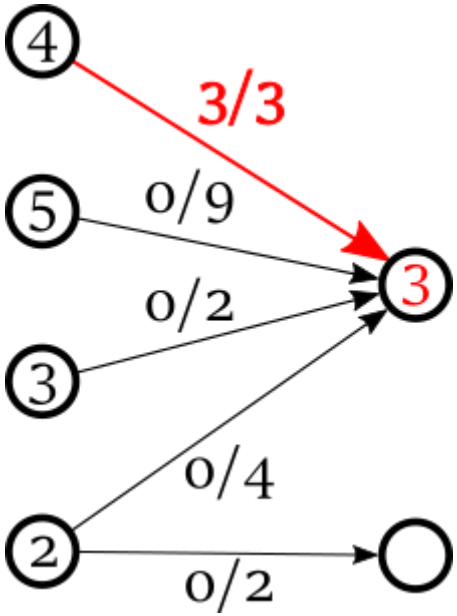
- Store promised flow $p: E \to R^*$

$$p(w,v) = \min(cap(w,v) - f(w,v), h(w))$$

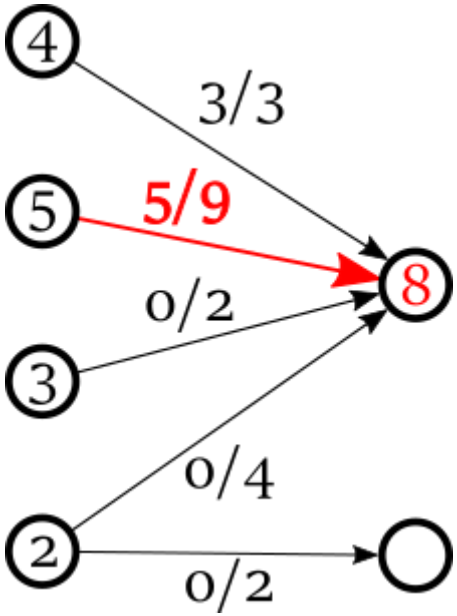
High Tide



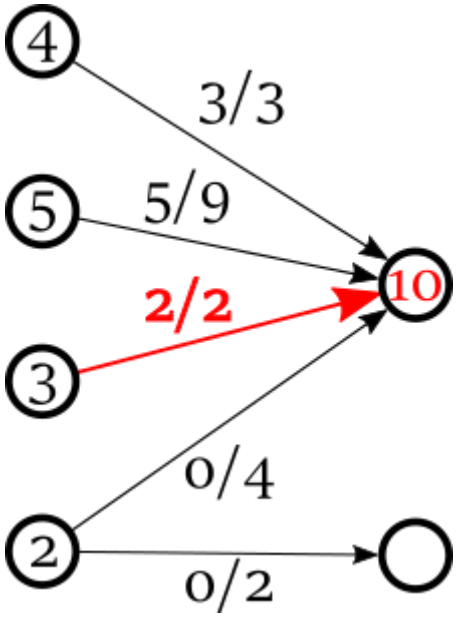
High Tide



High Tide



High Tide



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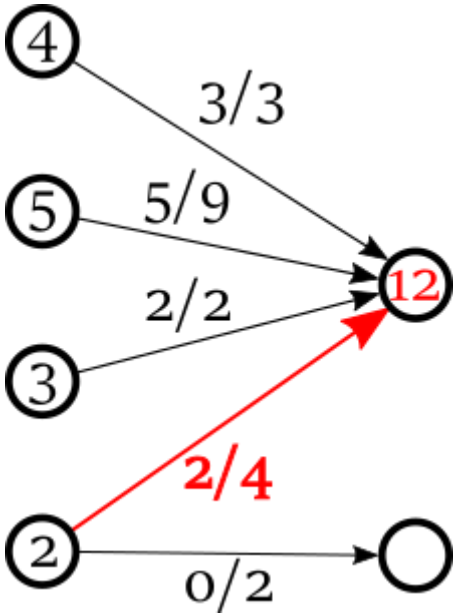

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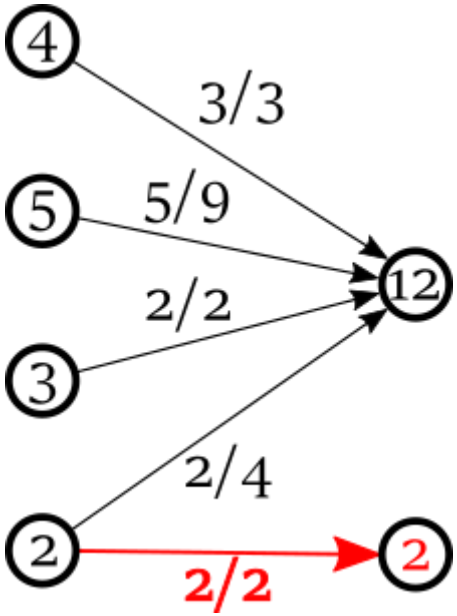

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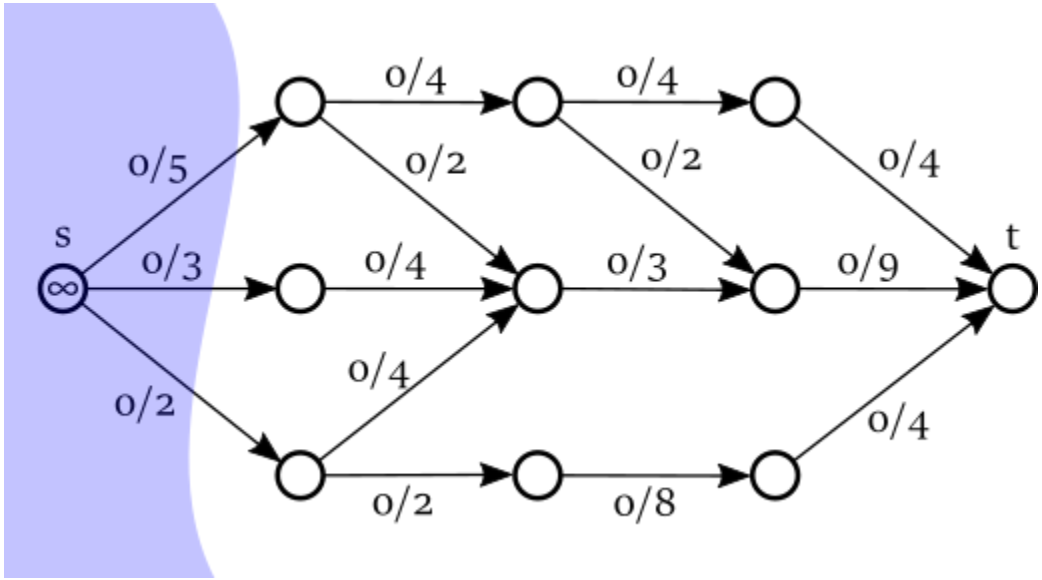

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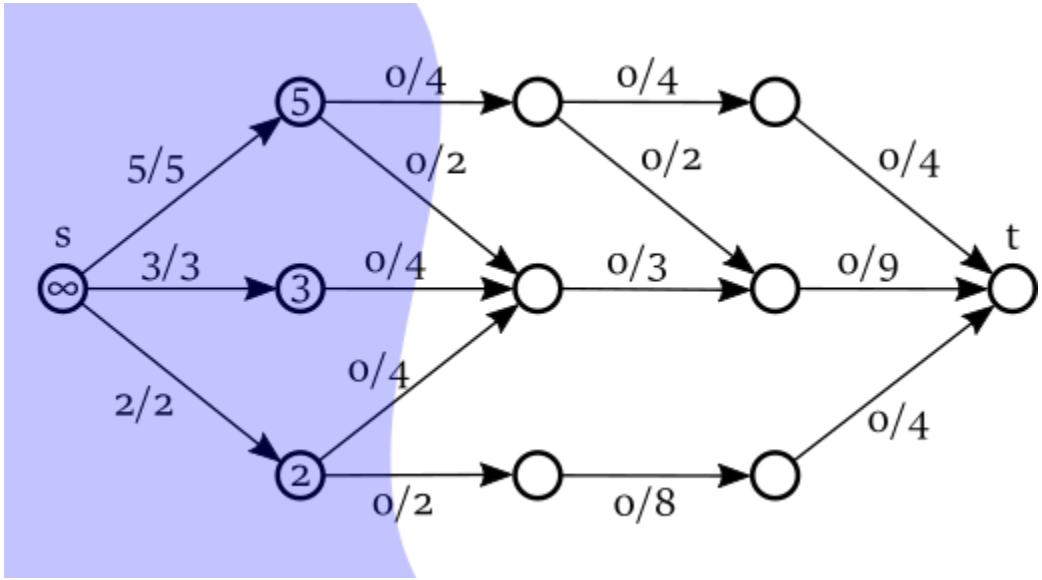

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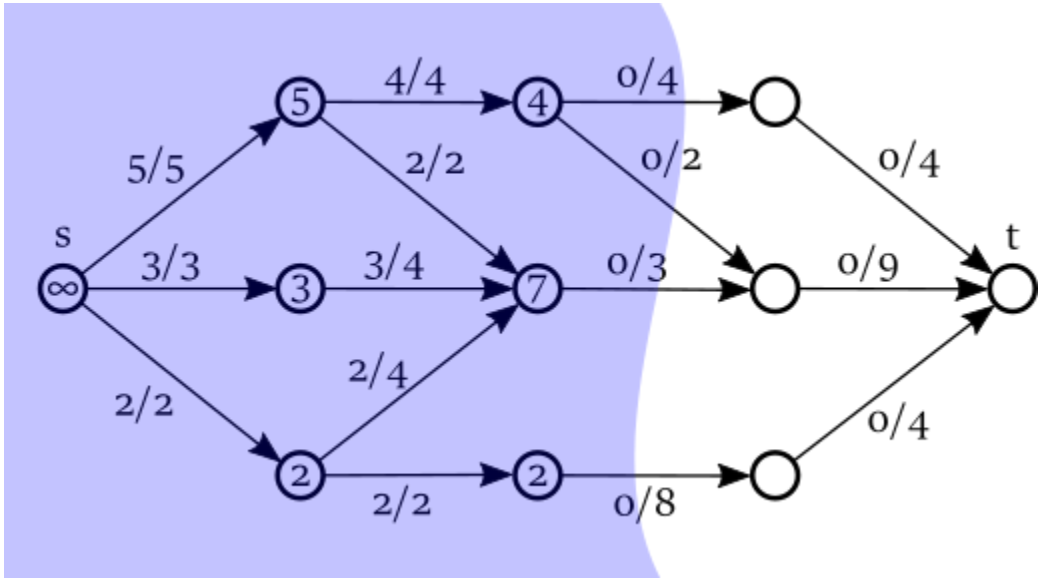

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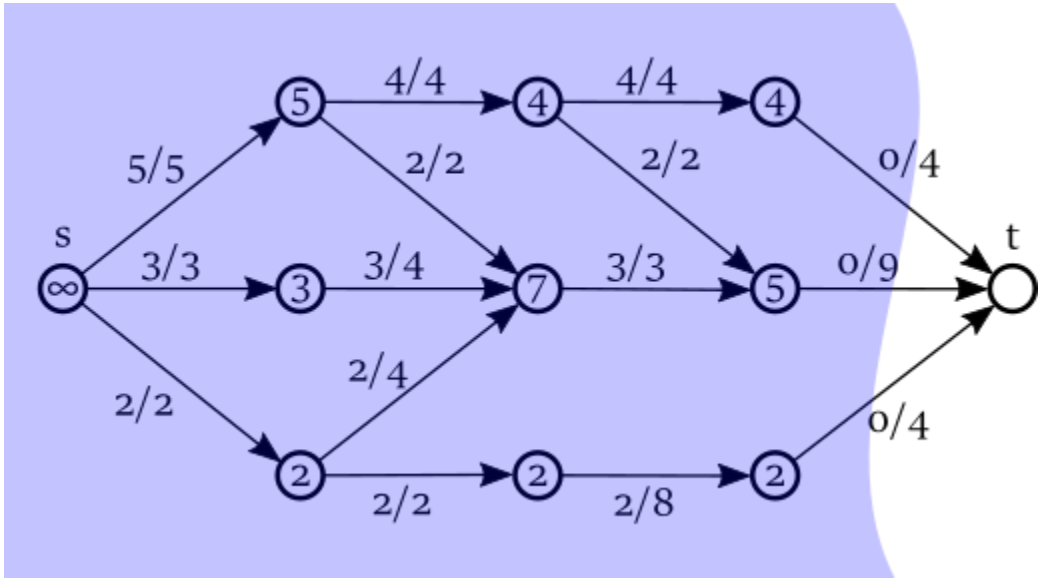

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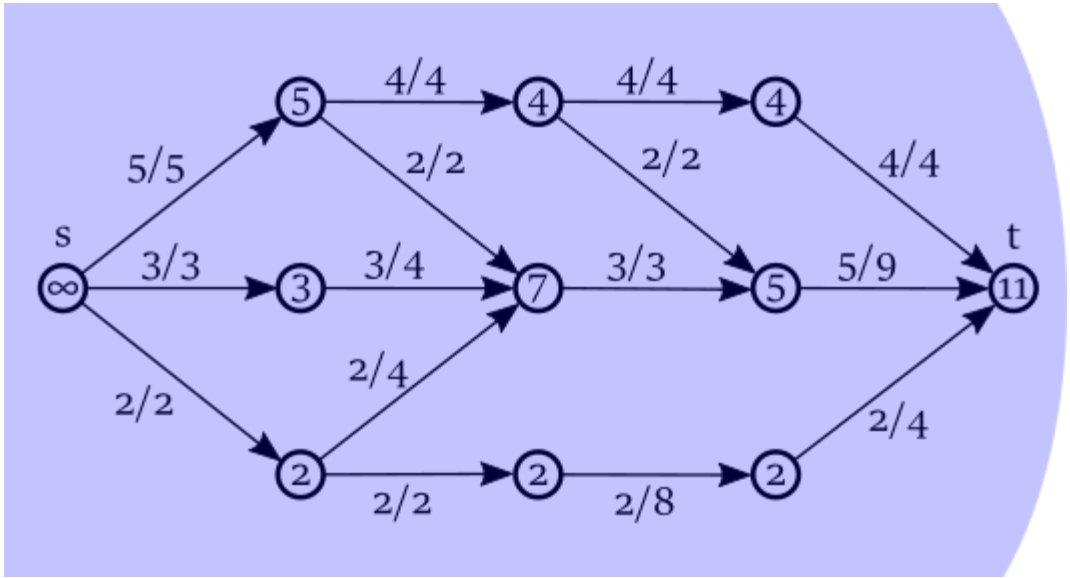
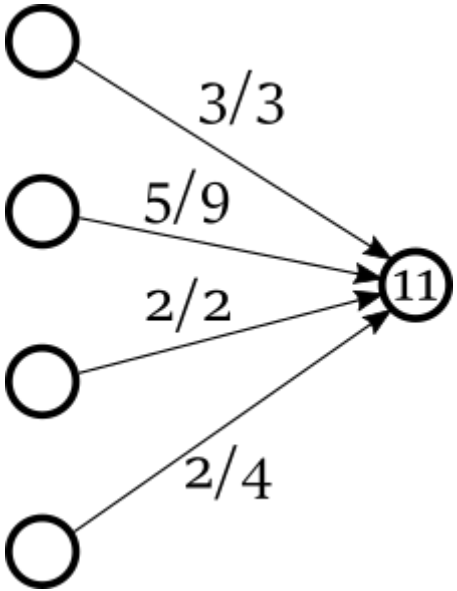


Low Tide



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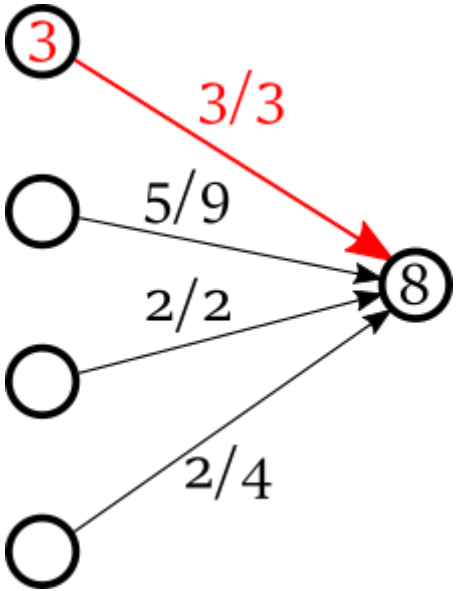

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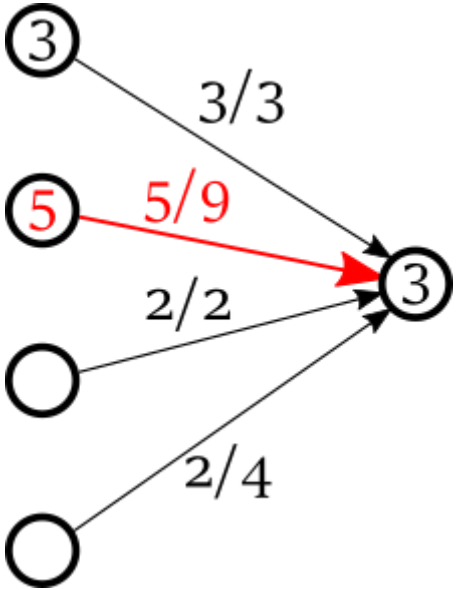
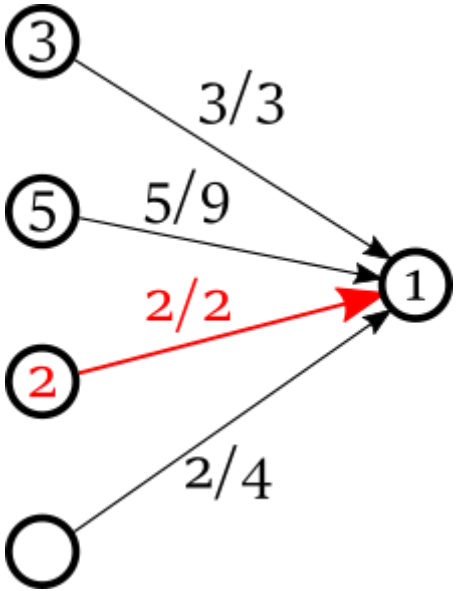
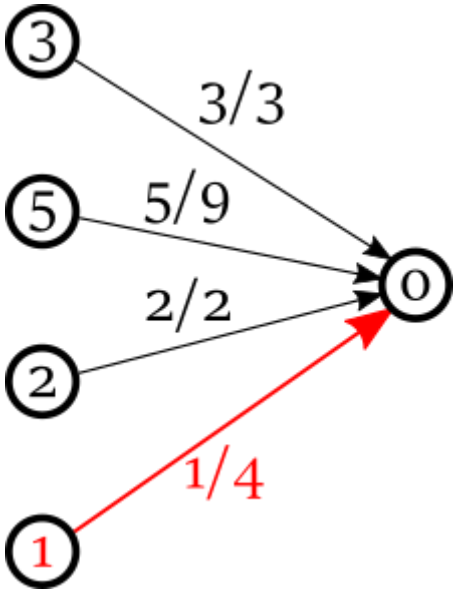


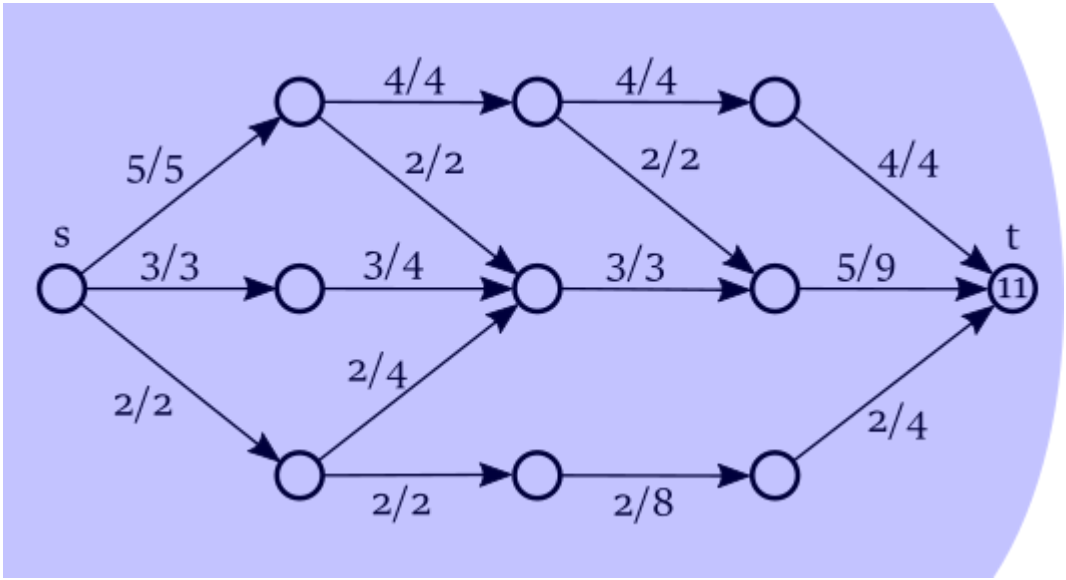
Low Tide



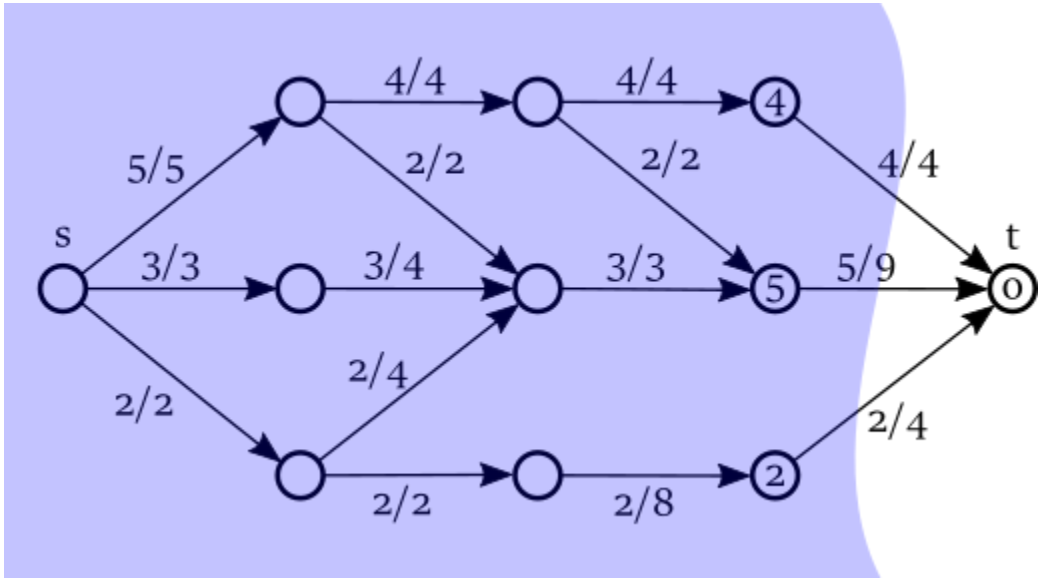
Low Tide



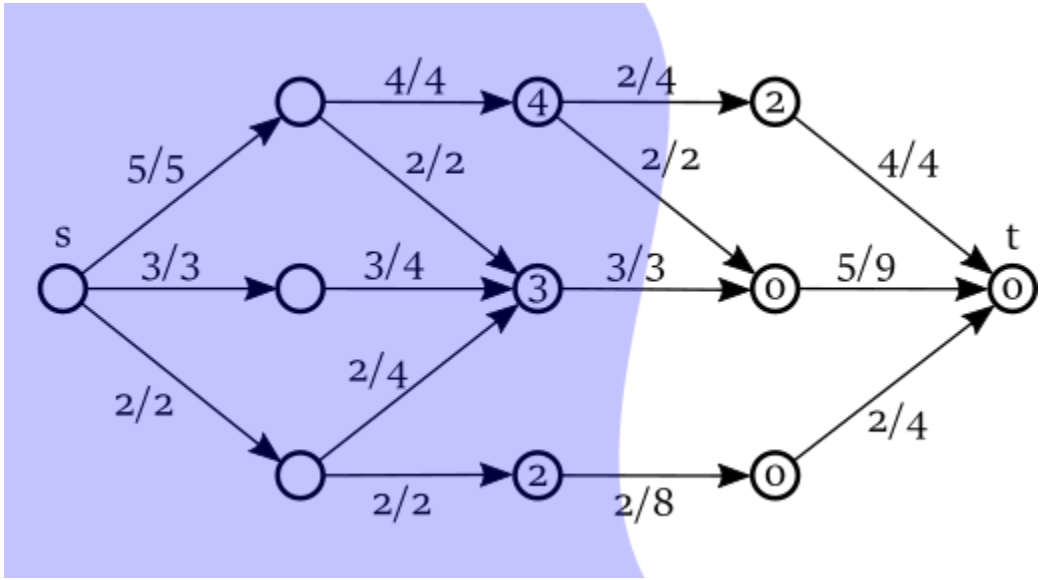
Low Tide



Low Tide



Low Tide



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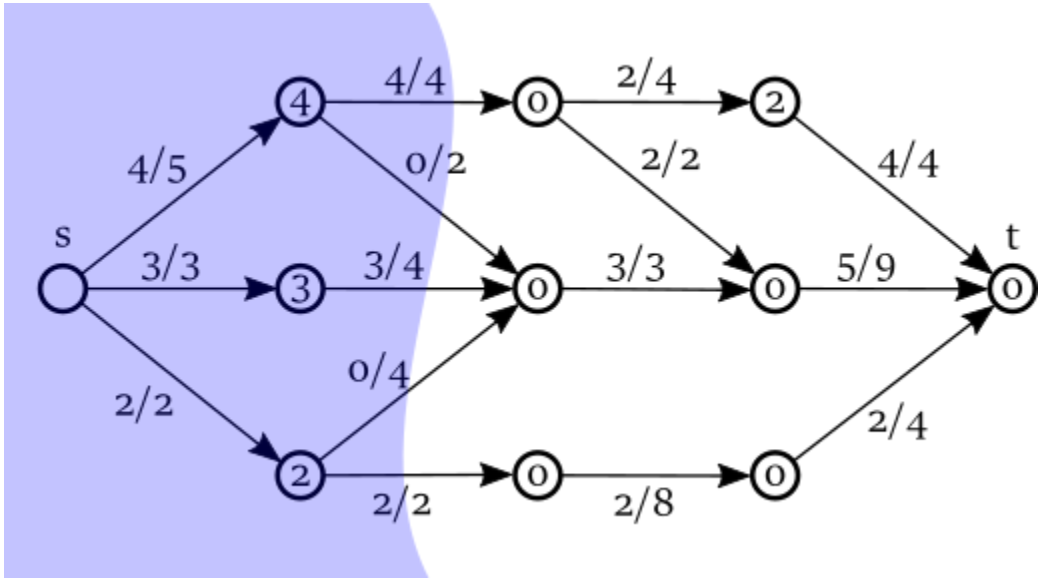
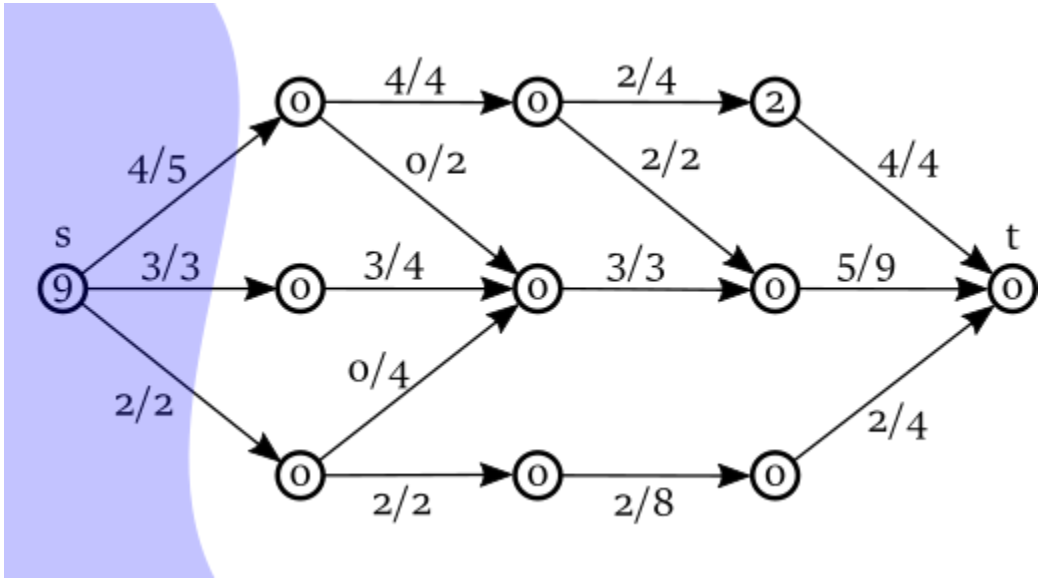
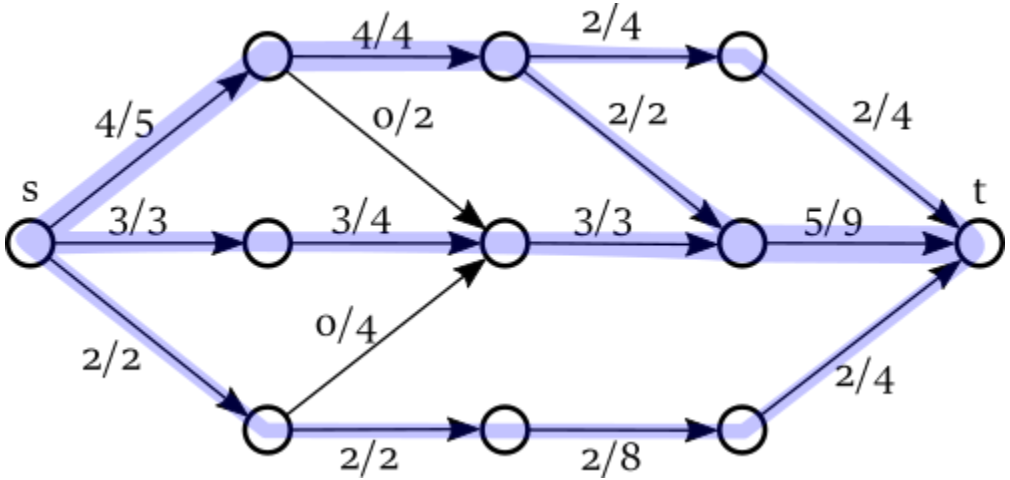


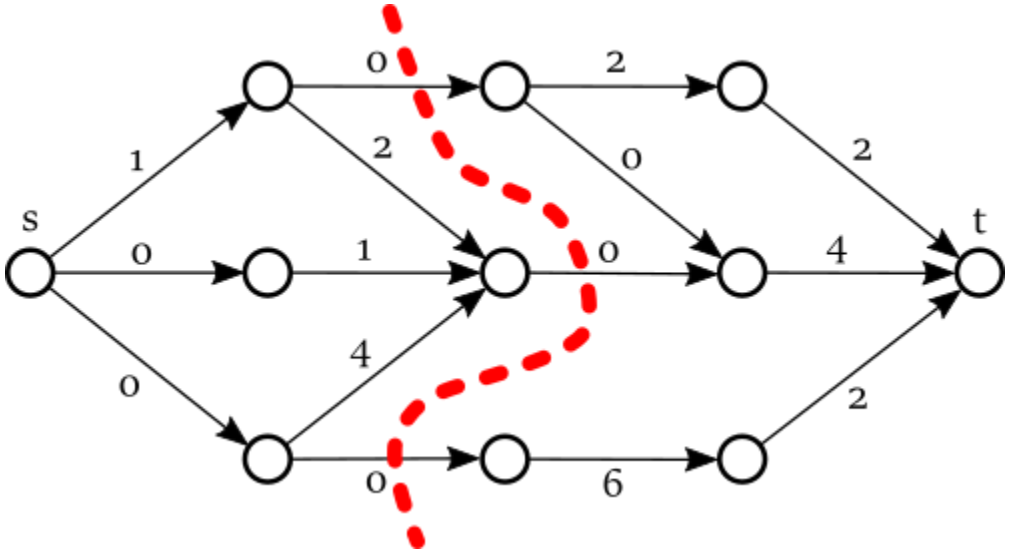
Low Tide



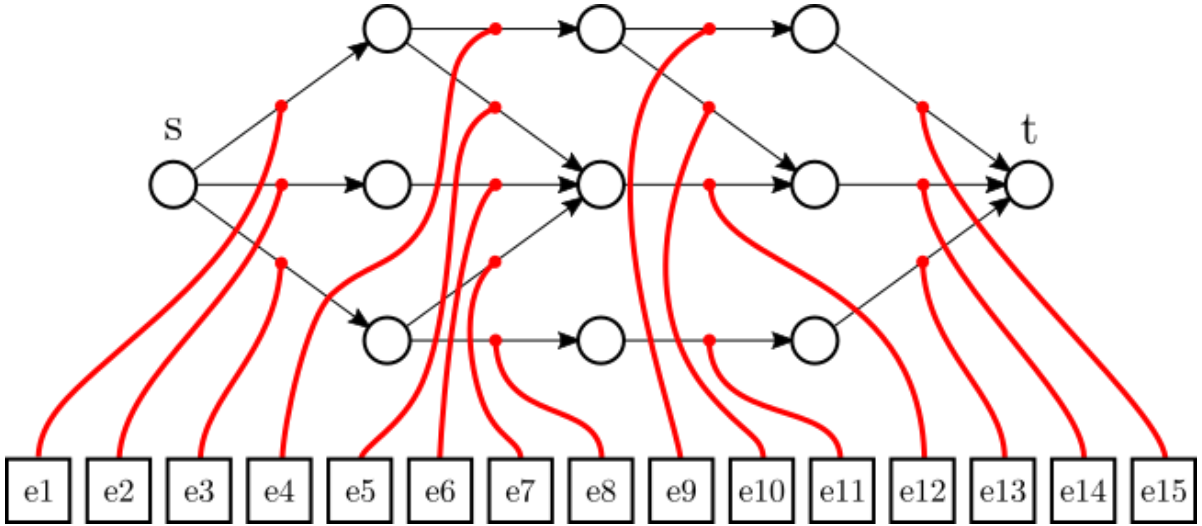
Erosion



Erosion



Implementation



Implementation: High Tide

$h(v) = 0$ for all $v \in V$;
$h(source) \longleftarrow \infty$;
foreach $edge\ e_i(w, v) \in E$ **do**
    $p(e_i) \longleftarrow \min(cap(e_i) - f(e_i), h(w))$;
    $h(v) \longleftarrow h(v) + p(e_i)$;
end

Implementation: Low Tide

$l(v) = 0$ for all $v \in V$
$l(sink) \longleftarrow h(sink)$;
foreach *edge* $e_i(w, v) \in E$ in reverse order **do**
    $p(e_i) \longleftarrow \min(p(e_i), h(w) - l(w), l(v))$;
    $l(v) \longleftarrow l(v) - p(e_i)$;
    $l(w) \longleftarrow l(w) + p(e_i)$;
end

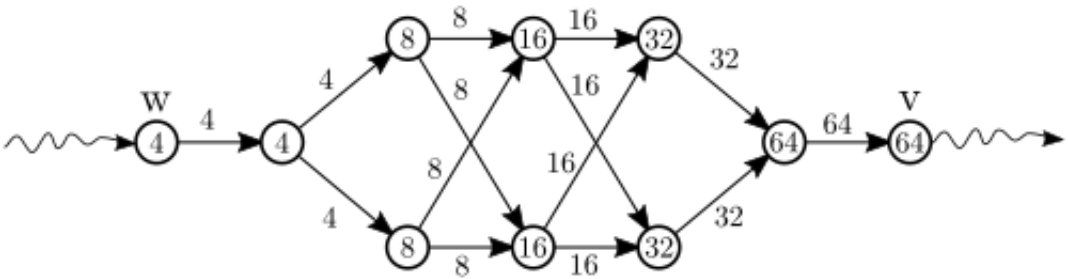
Implementation: Erosion

$h(v) = 0$ for all $v \in V$;
$h(source) \longleftarrow l(source)$;
foreach $edge\ e_i(w, v) \in E$ **do**
    $p(e_i) \longleftarrow \min(p(e_i), h(w))$;
    $h(w) \longleftarrow h(w) - p(e_i)$;
    $h(v) \longleftarrow h(v) + p(e_i)$;
    $f(e_i) \longleftarrow f(e_i) + p(e_i)$;
    $f\big(rev(e_i)\big) \longleftarrow f\big(rev(e_i)\big) - p(e_i)$;
end

Run-time Analysis

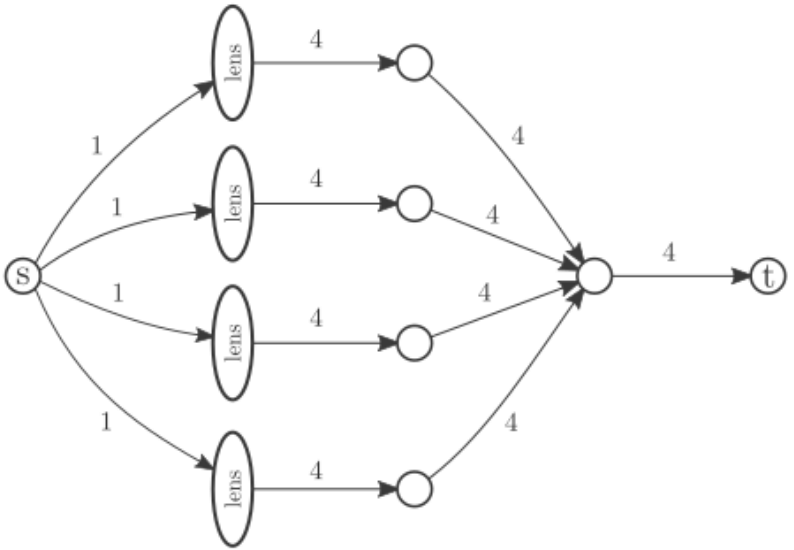
- Shortest Augmenting Path: $O(nm^2)$

Bounding Tide Cycles



Bounding Tide Cycles

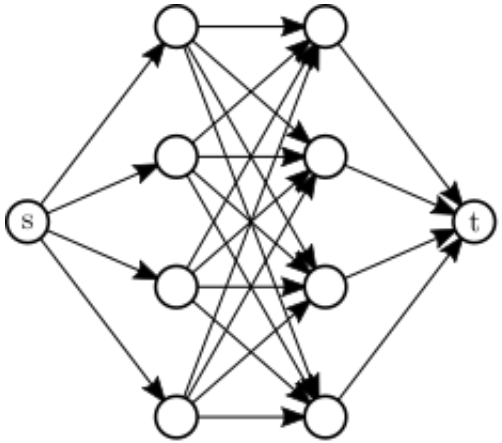
Requires $O\left(\frac{n}{\log(n)}\right)$ tide cycles.

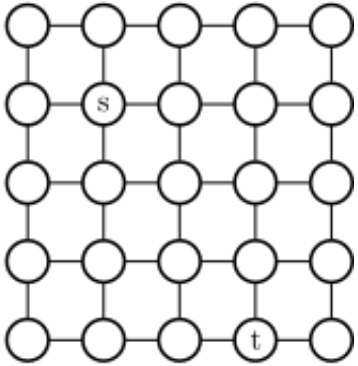


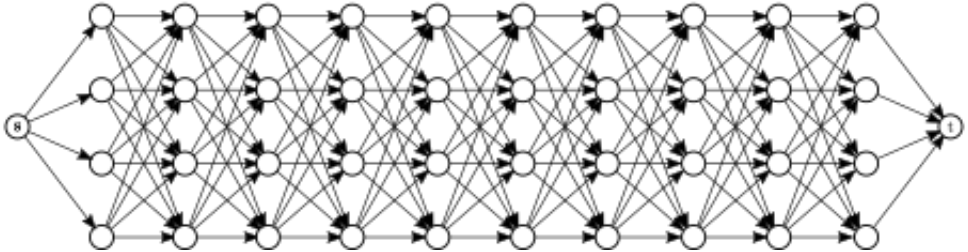
Experiments

- Six suites of tests
- 200 tests per suite (10 different values of n)
- Experiments run against five other algorithms
- Also included a version of Tidal Flow without heuristic function h
- Each algorithm run for a maximum of 20 seconds

Test Suites







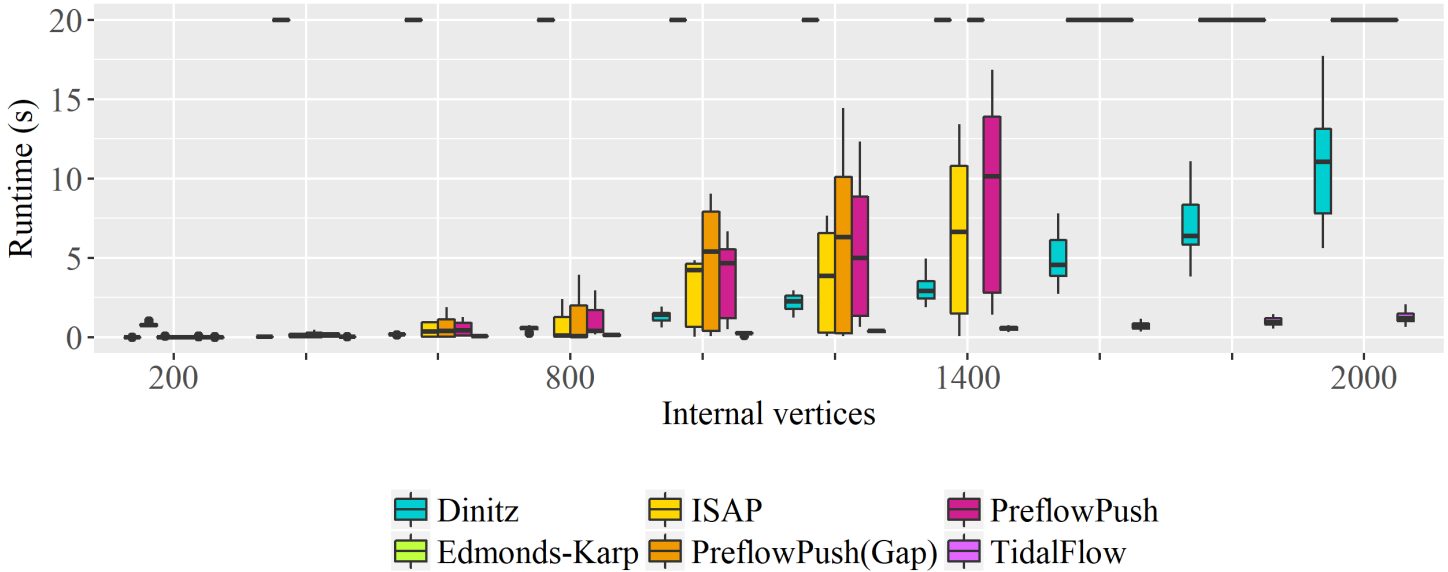
Bipartite Matching Suites

- Sparse versus dense
- Unit versus high capacity

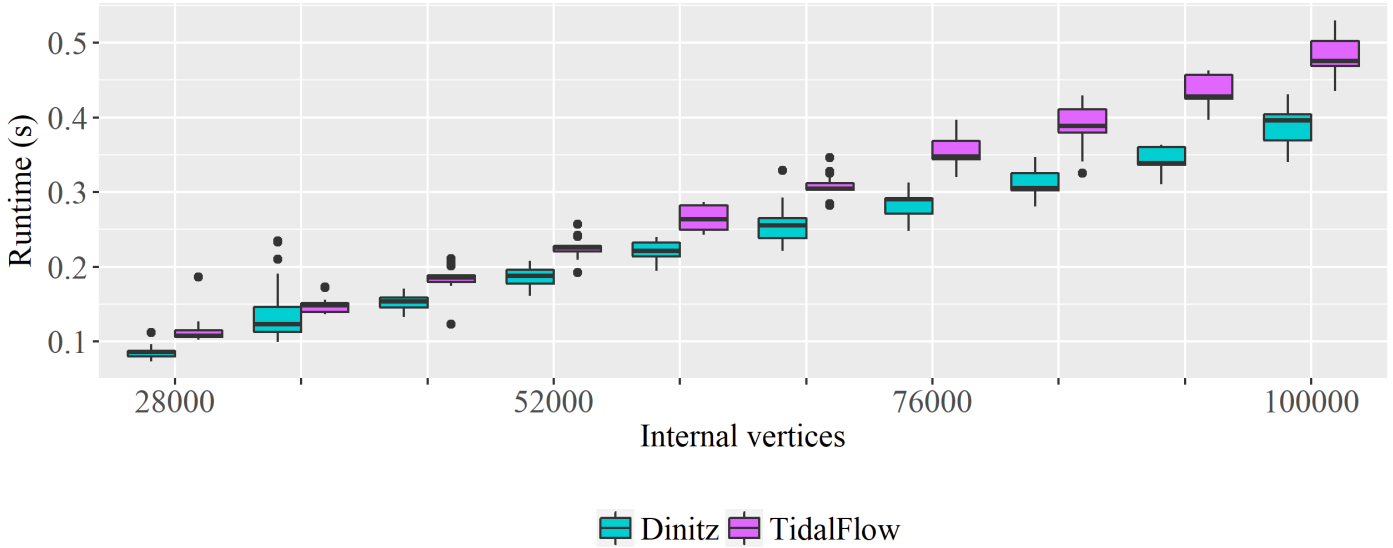
Flow Algorithms

- Shortest Augmenting Path (Edmonds and Karp, 1972)
- Dinitz Algorithm (Dinitz, 1970)
- Preflow-Push (Goldberg and Tarjan, 1988)
- Preflow-Push with Gap Heuristic (Goldberg and Tarjan, 1988)
- Improved Shortest Augmenting Path (Orlin and Ahuja, 1987)

Results: Dense High Capacity BPM



Results: Sparse Unit BPM



Open Questions

- Empirical study is only preliminary
- Bounding tide cycles: $O\left(\frac{n}{\log(n)}\right)$ versus $O(m)$
- Empirical results suggest $O(nm^2)$ runtime is not tight
- Does a better h exist?

Questions?