Math contests: solutions without solving

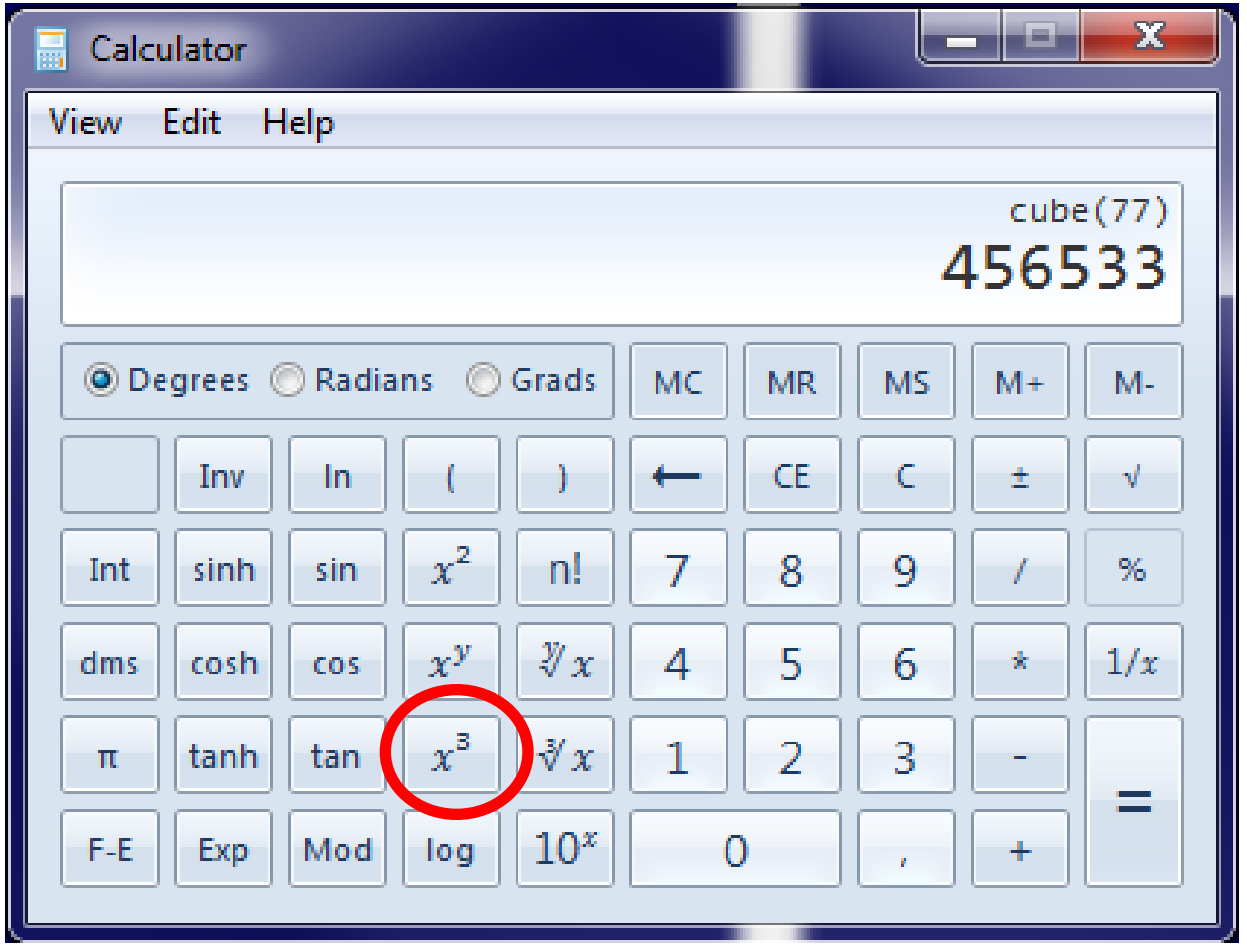
MĀRTIŅŠ OPMANIS
INSTITUTE OF MATHEMATICS AND COMPUTER SCIENCE,
UNIVERSITY OF LATVIA

IOI Conference, Almaty, July 28, 2015

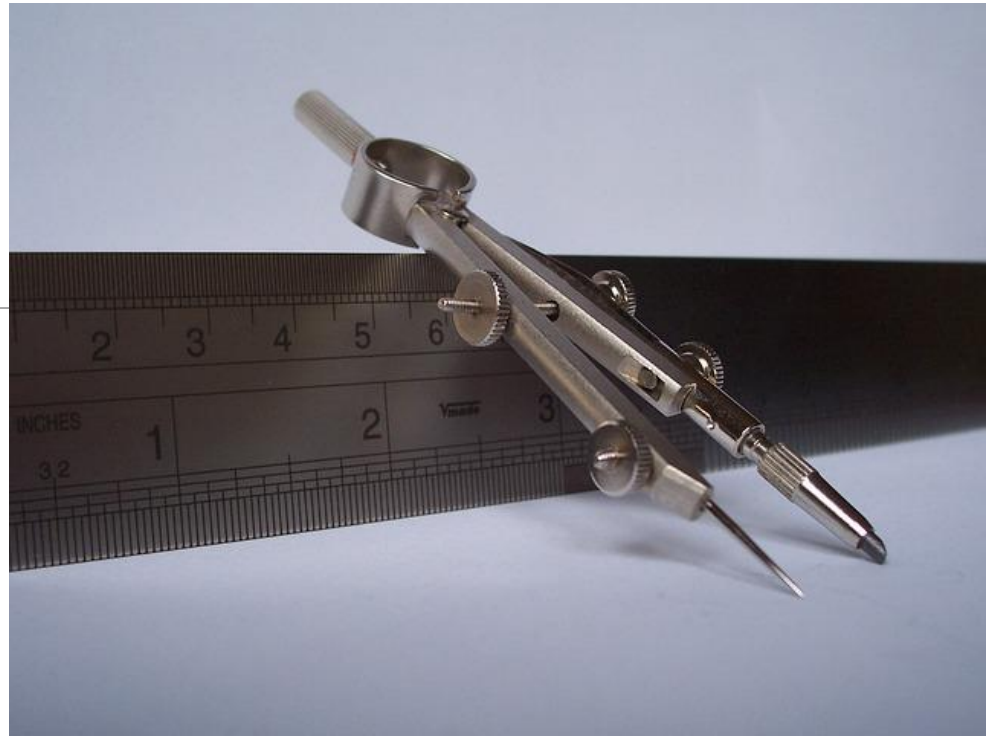
Simple math task: «Calculate 77^3»

1. Are you going to try this without additional tools?
2. Do you know appropriate tool to calculate this?
3. Can you use such tool?

Calculate 77^3



Tools …



- allowed in math contests
  - straightedge
    (ruler with no marks –
    try to find photo of such tool)
  - compas
- around us in everyday life

How confident you feel without …

- mobile phone
- online maps and navigator
- e-mail
- Internet

Q. What is the **solution** of Math contest problem?

A. Depends on contest format.

Classical «long form» solution

- contains proofs and reasoning,
- all necessary steps shown

IMO 2015

Problem 5. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x + y)\big) + f(xy) = x + f(x + y) + yf(x)$$

for all real numbers x and y.

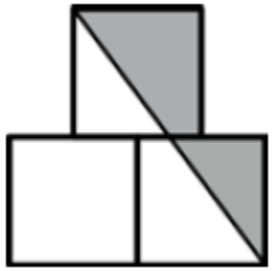
✓ Needs human resources to check correctness of solution

«Short form» solution – just correct answer

- usually as one from the few given choices

12. In the figure, each square has sides of length 1. What is the area of the shaded region?

(A) $\frac{3}{4}$ (B) $\frac{7}{8}$ (C) 1 (D) $1\frac{1}{4}$ (E) $1\frac{1}{2}$



Canadian Math Kangaroo Contest

- ✓ Easy to check and grade
- ✓ Lucky guessing possible

What will happen if tools and resources (T&R) like …

- Calculators
- Computers
- Internet

… would be allowed at mathematical contests?

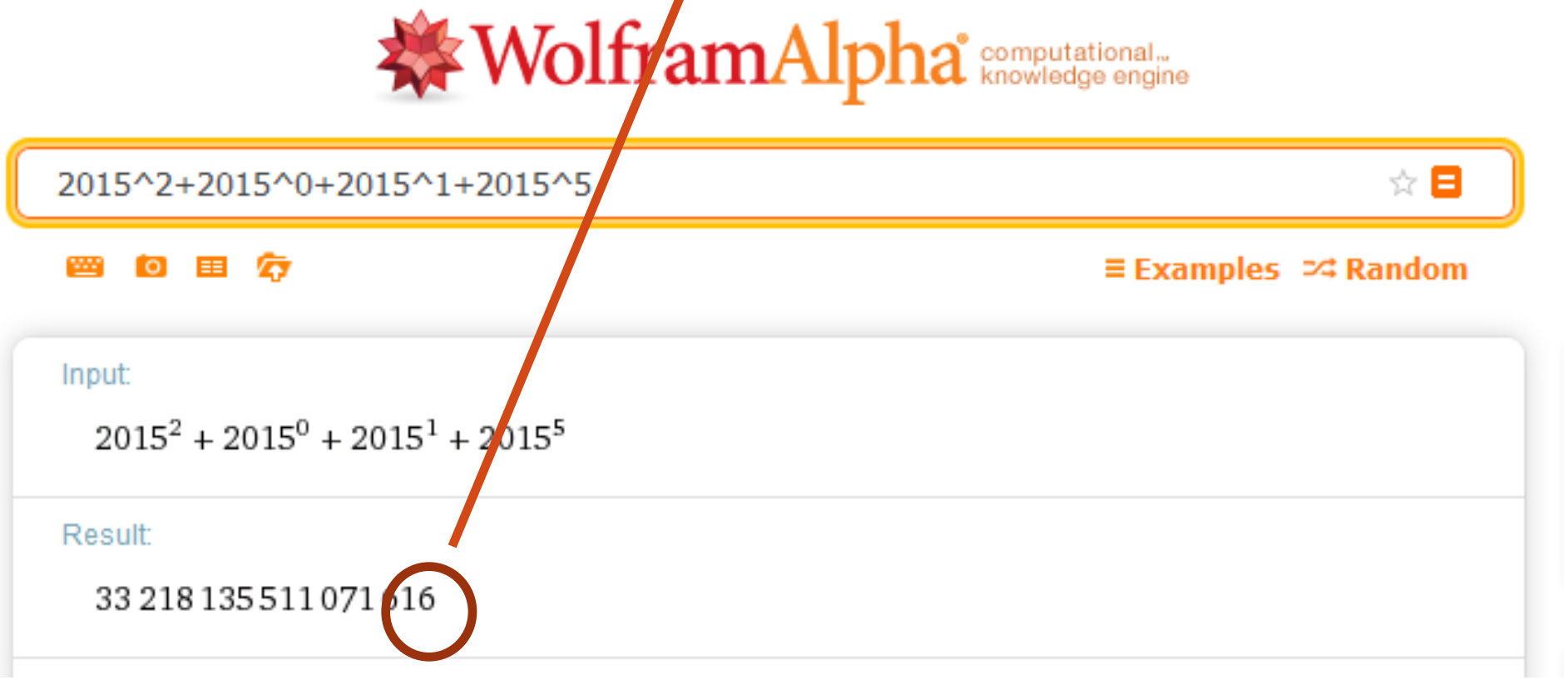
Frame of my experiment, also checklist for contest where T&R are planned

- No help from outside
- Only free R&T
- R&T must be useful to speed up finding of answer
- No possibility to use attribution of previously published task
- Tasks must be non-trivial and non-routine
- There must exist verifyable correct answer

What is the units digit of the number $2015^2 + 2015^0 + 2015^1 + 2015^5$?

(A) 1 (B) 5 (C) 6 (D) 7 (E) 9

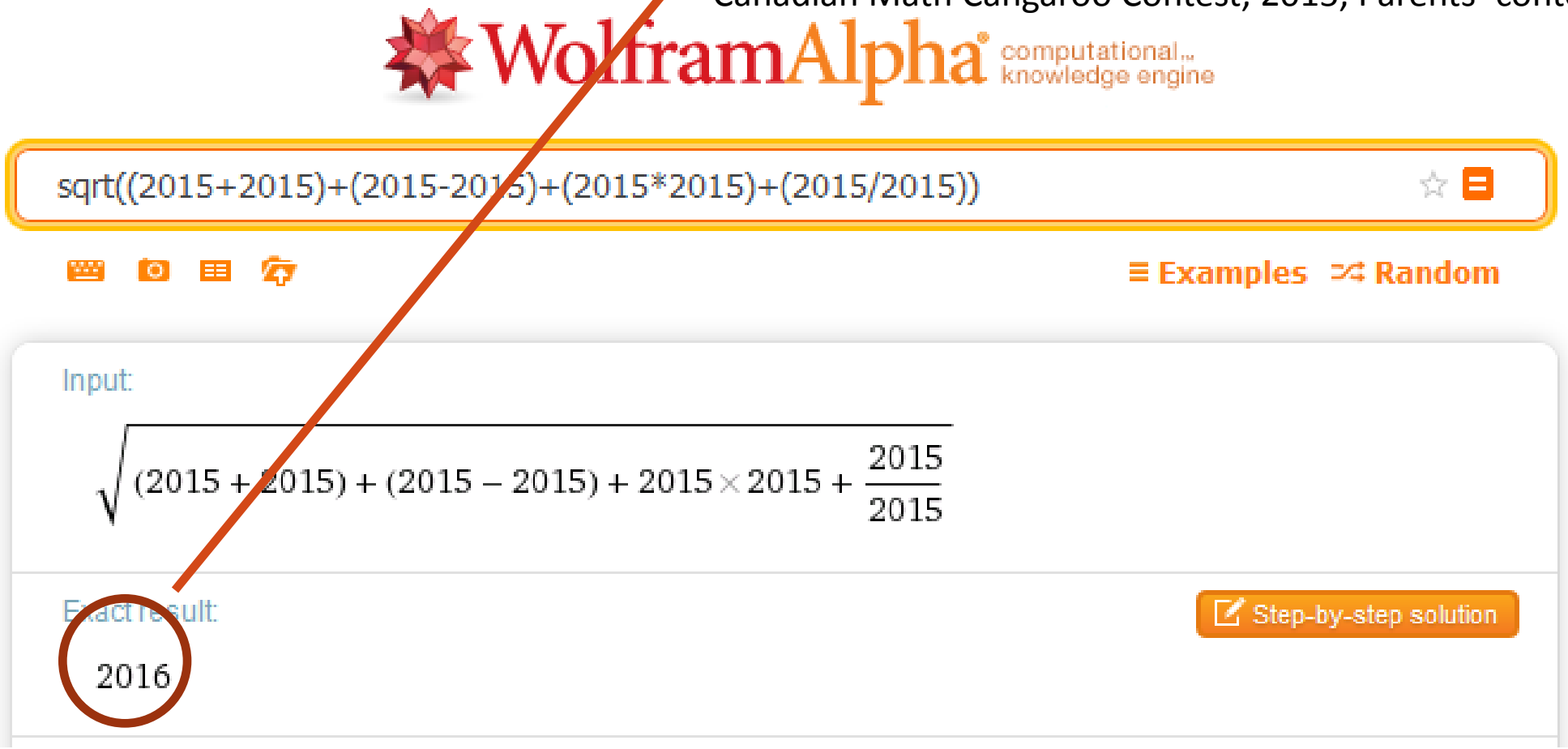
Canadian Math Cangaroo Contest, 2015, Parents' contest



$$\sqrt{(2015+2015)+(2015-2015)+(2015\cdot 2015)+(2015:2015)}$$

(A) $\sqrt{2015}$ (B) 2015 (C) 2016 (D) 2017 (E) 4030

Canadian Math Cangaroo Contest, 2015, Parents' contest



Task: «Calculate volume of dodecahedron with length of edge 1.»

1. Are you going to try this without additional tools?
2. Do you know appropriate tool to calculate this?
3. Can you use such tool?

Tīmeklis Attēli Video Maps Vēl Meklēšanas rīki

Aptuveni 151 000 rezultāti (0,47 sekundes)

Tiek rādīti rezultāti vaicājumam volume of ***dodecahedron***
Labāk meklēt volume of dodecadron

What's the Volume of a Dodecahedron?
www.shlomifish.org › Puzzles › Math-Ventures ▾ Tulkot šo lapu
Calculates the **volume** of a **Dodecahedron**, a Platonic solid with 12 identical sides that are themselves perfect pentagons.

Dodecahedron -- from Wolfram MathWorld
mathworld.wolfram.com › ... › Polyhedra › Platonic Solids ▾ Tulkot šo lapu
autors EW Weisstein - 2003 - Minēts 2 - Saistītie raksti

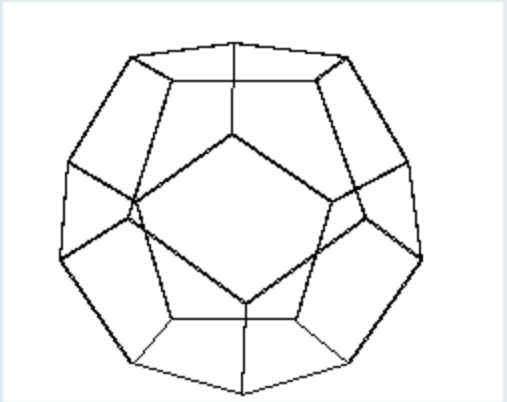
- Shlomi Fish's Homepage
- About Myself
  - Bio
  - Contact Me
  - My Résumés
  - Personal Ad
  - My Weblogs
  - Interviews
  - "Rindolf" - my nickname
- Humour
  - Stories
  - Aphorisms and Quotes
  - Small Scale
  - By Others
  - Recommendations
- Essays
  - Index to Essays
  - Book Reviews
  - Philosophy

If you like this site, see my recent campaign for collecting money for my works of fiction.

Puzzles Section Menu

Show

What's the Volume of a Dodecahedron?



The picture above shows a dodecahedron. It's a solid body which has 12 perfect pentagons of the same size as its sides. I wondered what is the volume of such a body.

What is the volume of a dodecahedron with an edge-length of a?

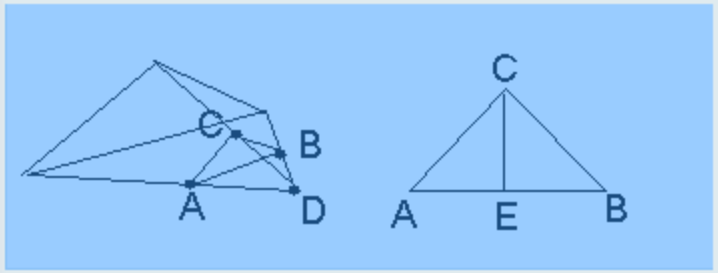
The solution can be found some space below:

Solution:

Since a dodecahedron is a perfect solid with 12 identical sides at the same distance from its centre, we can divide it into twelve identical pyramids. The corner of each pyramid is found at the dodecahedron's centre, and the base is one of the sides.

The volume of a pyramid is the area of the base multiplied by its height divided by 3. We can calculate the height by the angle between the base and one of the surface sides. Since two pyramids are adjacent at every edge of the dodecahedron, we can find it by taking the angle between two adjacent sides of the dodecahedron and dividing it by 2.

To find that, let's take a corner of the dodecahedron, and form a triangle at the points which are at a certain distance along the edges. We get the following picture:



From point C, which is found somewhere along the edge, let's lower two perpendiculars to the edge, down to the edges of the base triangle. We get the triangle CAB. Now since the surface sides of this pyramid are isosceles triangles, and the angle of a perfect pentagon is equal to:

$$\frac{180^\circ \cdot 3}{5} = 108^\circ$$

The volume of the pyramid is the area of its base multiplied by its height (OD) divided by 3 and so it is equal to:

$$\frac{5}{4} \cdot \tan 54^\circ \cdot \mathbf{a}^2 \cdot$$
$$* \frac{1}{2} \cdot tan 54^\circ \cdot \tan\left[\arcsin\left(\frac{1}{2 \sin 36^\circ}\right)\right] \cdot \mathbf{a} \cdot \frac{1}{3} =$$
$$* \frac{5}{24} \cdot \tan^2 54^\circ \cdot \tan\left[\arcsin\left(\frac{1}{2 \sin 36^\circ}\right)\right] \cdot \mathbf{a}^3$$

Since there are 12 such pyramids in a dodecahedron, its volume is equal to this volume multiplied by 12.
Thus, we get that the volume of a dodecahedron is:

$$\frac{5}{2} \cdot \tan 54^{\circ 2} \cdot \tan\left[\arcsin\left(\frac{1}{2 \sin 36^\circ}\right)\right] \cdot \mathbf{a}^3 = 7.66 * \mathbf{a}^3.$$

Algebra
Applied Mathematics
Calculus and Analysis
Discrete Mathematics
Foundations of Mathematics
Geometry
History and Terminology
Number Theory
Probability and Statistics
Recreational Mathematics
Topology
Alphabetical Index

Dodecahedron

DOWNLOAD Mathematica Notebook
EXPLORE THIS TOPIC IN The MathWorld Classroom

The area of a single face is the area of a pentagon of unit edge length

$$A = \frac{1}{4}\sqrt{25 + 10\sqrt{5}},$$

so the surface area is 12 times this value, namely

$$S = 3\sqrt{25 + 10\sqrt{5}}.$$

The volume of the dodecahedron can be computed by summing the volume of the 12 constituent pentagonal pyramids,

$$V = 12\left(\frac{1}{3}A\,r\right) = \frac{1}{4}\left(15 + 7\sqrt{5}\right).$$

Apollonius showed that for an icosahedron and a dodecahedron with the same inradius,

$$\frac{V_{\text{icosahedron}}}{V_{\text{dodecahedron}}} = \frac{A_{\text{icosahedron}}}{A_{\text{dodecahedron}}},$$

where V is the volume and A the surface area, with the actual ratio being

Where is the problem?

$$x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 2002$$

WolframAlpha computational knowledge engine

x*floor(x*floor(x*floor(x)))=2002

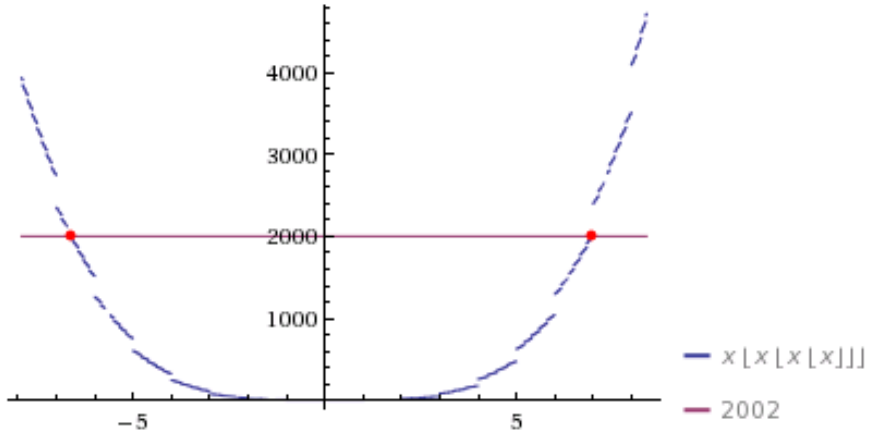
Examples Random

Input:

$$x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 2002$$

$\lfloor x \rfloor$ is the floor function

Plot:



Numerical solution:

$$x = 7.$$

Download page

POWERED BY THE WOLFRAM LANGUAGE

7*floor(7*floor(7*floor(7)))

Examples Random

Input:

$7\lfloor 7\lfloor 7\lfloor 7\rfloor\rfloor\rfloor$

$\lfloor x \rfloor$ is the floor function

Result:

2401 ≠2002

Things are not so bad, because…

WolframAlpha computational… knowledge engine

solve p^3-q^5=(p+q)^2 over integers

Examples Random

Input interpretation:

solve | $p^3 - q^5 = (p+q)^2$ | over the integers

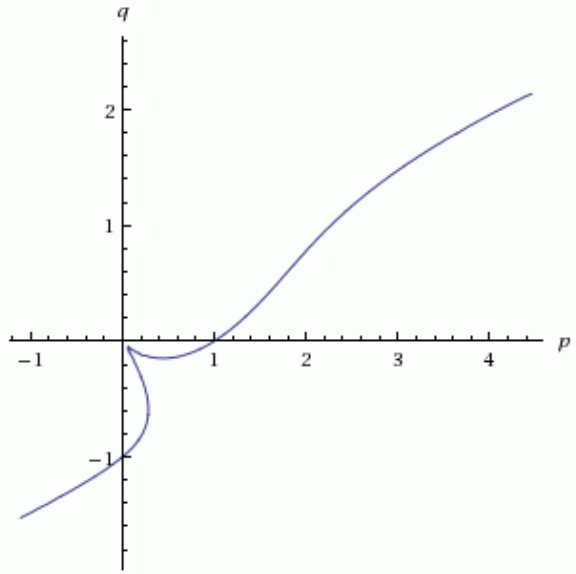
Results:

$q = -1 \land p = 0$

$q = 0 \land p = 1$

$e_1 \land e_2 \land \ldots$ is the logical AND function »

Implicit plot:



Enable interactivity

Computed by Wolfram *Mathematica*

Download page

WolframAlpha computational… knowledge engine

solve p^3-q^5=(p+q)^2 over integers

Examples Random

Input interpretation:

solve | $p^3 - q^5 = (p+q)^2$ | over the integers

Results:

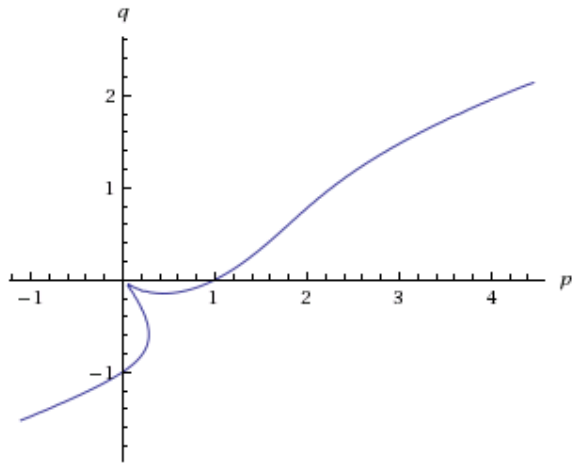
$p = 0$ and $q = -1$

$p = 0$ and $q = 0$

$p = 1$ and $q = 0$

$p = 7$ and $q = 3$

Implicit plot:



Enable interactivity

Download page

POWERED BY THE WOLFRAM LANGUAGE

Conclusions

- T&R may be allowed but contests will look quite different
- Several types of tasks will not survive
- Without T&R some tasks may be too difficult to solve, i.e. expertise in T&R becomes requirement
- Instead of «just finding» additional time must be spent for checking and rechecking of obtained results

Thank you!

Questions?